Advanced Modern Macroeconomics

Dynamic $AS - AD$ Solution Methodology

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- Chapter shows analytic solution to dynamic economy,
  - with zero growth, stationary balanced growth equilibrium.
  - All equilibrium conditions used find equilibrium.
  - All variables determined simultaneously,
  - any one variable can be solved for initially
  - then it is used to solve all other variables.
- Capital stock as state variable solved initially.
  - with closed form solution.
  - So an explicit function of given exogenous parameters.
  - Comparative static change of parameter
  - gives new capital stock using closed form solution.

Summary

- Solves for capital stock in two related ways.
  - First: Use *AS*-*AD*, giving one equation in two unknowns:
  - capital stock, wage rate.
  - Use exogenous growth to solve for equilibrium capital/labor.
  - Solve wage rate using capital/labor,
  - substitute into $AS - AD$, get one equation in k_t.
- Second: just use all equilibrium conditions: same k_t.
- Graphically solve for capital stock in four-quadrant graph.

Building on the Last Chapters

- Derives k_t for all Chapters 8, 9 examples.
- Adds new comparative static experiments.
- Separate chapter to focus on solution methodology.
- Uses $AS - AD$ in crucial way: to solve k_t.
- Develops new graphical analysis using $AS - AD$
  - expanding equilibrium to four quadrants
  - so capital stock itself can be shown graphically.
  - gives full analysis within a graph with output market.
- Graphs are significant extension beyond $AS - AD$
  - uses two main margins of model: consumer, firm.
  - Provided for baseline dynamic example
  - and comparative static increase in goods productivity,
  - to illustrate idea that capital stock can be found graphically.

Learning Objective

- Realization that all equilibrium conditions must be used.
- Trick provided: use $AS - AD$ analysis and get a short-cut,
  - since already one equation in two unknowns.
- Must be only one equation in just one unknown: k_t here.
- So exploit exogenous zero growth assumption
  - solve for wage rate, substitute w_t into $AS - AD$;
  - one equation in just k_t;
  - and then solve k_t explicitly.
- See that can modify a traditional textbook approach of
  - 4-quadrant graph to solve entire economy,
  - including capital.

Who Made It Happen

- Frank Ramsey: Dynamic equilibrium model.
- Solving model: Sargent's 1987 *Dynamic Macroeconomic Theory*,
- Stokey, Lucas, Prescott, 1898 *Recursive Methods in Economic Dynamics*
- Cooley 1995 edited *Frontiers in Business Cycle Research*
  - clarifies dynamic methodology and application.
- Ljungqvist, Sargent 2004 *Recursive Macroeconomic Theory.*
- Ohanian et al. 2009: overview of dynamic general equilibrium theory.

Full Model Solution to Derive Capital Stock

- Exploit zero exogenous growth assumption

$$1 = \frac{c_{t+1}}{c_t} = \frac{1 + r_t - \delta_k}{1 + \rho}, \quad (1)$$

$$\Longrightarrow \quad r_t = \rho + \delta_k. \quad (2)$$

- plus marginal product of capital, labor: solve w_t;

$$\rho + \delta_k = r_t = (1 - \gamma) A_G \left(\frac{l_t}{k_t} \right)^{\gamma}, \quad (3)$$

$$\Longrightarrow \quad \frac{l_t}{k_t} = \left[\frac{\rho + \delta_k}{(1 - \gamma) A_G} \right]^{\frac{1}{\gamma}}. \quad (4)$$

$$w_t = \gamma A_G \left(\frac{l_t}{k_t} \right)^{\gamma - 1},$$

$$\Longrightarrow \quad w_t = \gamma A_G \left[\frac{\rho + \delta_k}{(1 - \gamma) A_G} \right]^{\frac{\gamma - 1}{\gamma}}.$$

Using Excess Output Demand to Get Solution

$$
\begin{aligned}
0 &= y_t^d - y_t^s \equiv Y\left(w_t, k_t\right) \\
&= \frac{w_t T + k_t\left[\rho + (1+\alpha)\,\delta_k\right]}{1+\alpha} - A_G^{\frac{1}{1-\gamma}}\left(\frac{\gamma}{w_t}\right)^{\frac{\gamma}{1-\gamma}} k_t; \\
w_t &= \gamma A_G\left[\frac{\rho+\delta_k}{(1-\gamma)\,A_G}\right]^{\frac{\gamma-1}{\gamma}}. \\
0 &= \frac{\gamma A_G\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right]^{\frac{\gamma-1}{\gamma}} T + k_t\left[\rho + (1+\alpha)\,\delta_k\right]}{1+\alpha} \\
&\quad - A_G^{\frac{1}{1-\gamma}}\left(\frac{(\gamma)^{\frac{\gamma}{1-\gamma}}}{(\gamma A_G)^{\frac{\gamma}{1-\gamma}}\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right]^{\frac{-\gamma}{\gamma}}}\right) k_t.
\end{aligned}
$$

- Is one equation in one unknown k_t.
- k_t can be solved explicitly as function of exogenous parameters.

Explicit Analytic Solution for Capital Stock

$$k_t \left[1 - \frac{\left[\rho + (1+\alpha)\,\delta_k\right]}{(1+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)} \right] = \frac{\gamma A_G \left(\frac{\rho+\delta_k}{(1-\gamma)A_G}\right)^{\frac{\gamma-1}{\gamma}} T}{(1+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)};$$

$$k_t = \frac{\frac{\gamma A_G \left(\frac{\rho+\delta_k}{(1-\gamma)A_G}\right)^{\frac{\gamma-1}{\gamma}} T}{(1+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)}}{1 - \frac{\left[\rho+(1+\alpha)\delta_k\right]}{(1+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)}} = \frac{T\gamma A_G \left(\frac{\rho+\delta_k}{(1-\gamma)A_G}\right)^{\frac{\gamma-1}{\gamma}}}{(1+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \left[\rho + (1+\alpha)\,\delta_k\right]},$$

$$= \frac{T\gamma A_G^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \alpha\delta_k}.$$

- Transition dynamics excluded: only stationary equilibrium.
- All other variable solutions follow once k_t is solved.

Alternative Solution Approach

$$\rho + \delta_k = r_t = (1-\gamma) A_G \left(\frac{l_t}{k_t}\right)^{\gamma}; \Rightarrow \frac{l_t}{k_t} = \left[\frac{\rho + \delta_k}{(1-\gamma) A_G}\right]^{\frac{1}{\gamma}},$$

$$y_t = A_G \left(\frac{l_t}{k_t}\right)^{\gamma} k_t;\ c_t = y_t - i_t = y_t - k_{t+1} + k_t (1 - \delta_k)$$

$$c_t = A_G \left(\frac{l_t}{k_t}\right)^{\gamma} k_t - \delta_k k_t;\ w_t = \gamma A_G^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho + \delta_k}\right]^{\frac{1-\gamma}{\gamma}};$$

$$T - l_t = \frac{\alpha c_t}{w_t} = \frac{\alpha \left[A_G \left(\frac{l_t}{k_t}\right)^{\gamma} k_t - \delta_k k_t\right]}{\gamma A_G^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}};$$

$$T - \frac{\alpha k_t \left\{A_G \left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right] - \delta_k\right\}}{\gamma A_G^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}} = l_t = k_t \left[\frac{\rho + \delta_k}{(1-\gamma) A_G}\right]^{\frac{1}{\gamma}}.$$

$$
\begin{aligned}
k_t &= k = \frac{T}{\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right]^{\frac{1}{\gamma}} + \frac{\alpha\left\{A_G\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right]-\delta_k\right\}}{\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}} \\
&= \frac{T\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right]^{\frac{1}{\gamma}} + \alpha\left(\frac{\rho+\delta_k}{(1-\gamma)} - \delta_k\right)} \\
&= \frac{T\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{\gamma\left[\frac{\rho+\delta_k}{(1-\gamma)}\right]^{\frac{1-1+\gamma}{\gamma}} + \alpha\left(\frac{\rho+\delta_k}{(1-\gamma)} - \delta_k\right)} \\
&= \frac{T\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \alpha\delta_k}. \qquad (5)
\end{aligned}
$$

All Other Variables as Functions of Capital Stock

$$r_t = \rho + \delta_k ; \frac{l_t}{k_t} = \left[\frac{\rho + \delta_k}{(1-\gamma) A_G} \right]^{\frac{1}{\gamma}} ; \; w_t = \gamma A_G \left[\frac{(1-\gamma) A_G}{\rho + \delta_k} \right]^{\frac{1-\gamma}{\gamma}} ;$$

$$y_t = A_G \left[\frac{\rho + \delta_k}{(1-\gamma) A_G} \right] k_t ; \; i_t = \delta_k k_t ; \; c_t = A_G \left[\frac{\rho + \delta_k}{(1-\gamma) A_G} \right] k_t - \delta_k k_t .$$

$$x_t = T - l = T - \left[\frac{\rho + \delta_k}{(1-\gamma) A_G} \right]^{\frac{1}{\gamma}} k_t ;$$

$$x_t = \frac{\alpha c_t}{w_t} = \frac{\alpha \left(A_G \left[\frac{\rho+\delta_k}{(1-\gamma)A_G} \right] k_t - \delta_k k_t \right)}{\gamma \left(A_G \right)^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho+\delta_k} \right]^{\frac{1-\gamma}{\gamma}}} .$$

$$\Longrightarrow k_t = \frac{\alpha A_G \left[\frac{\rho+\delta_k}{(1-\gamma)A_G} \right] k_t - \alpha \delta_k k_t}{\gamma \left(A_G \right)^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho+\delta_k} \right]^{\frac{1-\gamma}{\gamma}} \left(T - \left[\frac{\rho+\delta_k}{(1-\gamma)A_G} \right]^{\frac{1}{\gamma}} \right)} .$$

Baseline Calibration: Target Values

- USA Post WWII values: work-leisure, c/y, r, k/y
- Broad approach for "target values":
  - really an illustration of economy in general,
  - rather than specification for specific country, time.
- Awake 16 hrs/day, 5 days/week; work 8 hrs/day,
  - Working time as percent of total time: $\frac{40}{16(5)} = \frac{1}{2}$.
  - ignores weekends: 16 hours free from two days;
  - Labor time: $\frac{40}{16(7)} = 0.357$.
  - But 2 hrs/day for maintenance with food and shopping,
  - 14 available hrs/day: working: $\frac{40}{14(7)} = 0.41$
  - 2 hrs/day taken up for our lifetime education,
  - 12 free hrs/day for work-leisure: working $\frac{40}{12(7)} = 0.48$,
- $l = 0.5$ is target; $x = 1 - l = 0.5$.
- $r = 0.06$ is plausible target value; $c/y = 2/3$ plausible..
- Use $\gamma < 0.5$ convex marginal cost; set $\gamma = \frac{1}{3}$.

Baseline Values, Targets; Examples 8.1, 9.1

$\rho = 0.03$, $\delta_k = 0.03$, $\gamma = \frac{1}{3}$, $T = 1$, $\alpha = 0.5$, $A_G = 0.15$.
$r_t = \rho + \delta_k = 0.03 + 0.03 = 0.06$; $k = 2.3148$:.

$$\begin{aligned} k &= \frac{T\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)-\alpha\delta_k}; \\ &= \frac{(1)\left(\frac{1}{3}\right)(0.15)^3\left(\frac{2}{3(0.06)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.06)(1.5)-0.5(0.03)} = 2.3148. \end{aligned}$$

$$l_t = k_t\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right]^{\frac{1}{\gamma}} = 2.3148\left(\frac{(0.06)\,1.5}{0.15}\right)^3 = 0.50000.$$

$$\begin{aligned} y_t &= A_G\left(\frac{l_t}{k_t}\right)^{\gamma}k_t = (0.15)\left(\frac{0.5}{2.314\,8}\right)^{\frac{1}{3}}(2.314\,8) = 0.20833; \\ k/y &= 2.3148/0.20833 = 11.11. \end{aligned}$$

Checking Consistency of Solution

$$
\begin{aligned}
0.06 &= \rho + \delta_k = r = (1-\gamma)\, A_G \left(\frac{l_t}{k_t}\right)^{\gamma} \\
&= \left(\frac{2}{3}\right)(0.15)\left(\frac{0.5}{2.3148}\right)^{\frac{1}{3}} = 0.06.
\end{aligned}
$$

$$
\begin{aligned}
\textit{demand} &: \\
w_t &= \gamma A_G \left(\frac{k}{l}\right)^{1-\gamma} = \left(\frac{1}{3}\right)(0.15)\left(\frac{2.314\,8}{0.5}\right)^{\frac{2}{3}} = 0.13889.
\end{aligned}
$$

$$
\begin{aligned}
\textit{supply} &: \\
w_t &= \frac{\alpha\rho k_t}{\left(T - l_t^s\right)(1+\alpha) - \alpha T} = \frac{0.5\,(0.03)\,2.3148}{(1-0.5)\,(1.5) - 0.5} = 0.13889.
\end{aligned}
$$

Calibration of Full Solution

$$k = 2.314\,8;\ \ l_t = 2.314\,8\left(\frac{(0.06)\,1.5}{0.15}\right)^3 = 0.50000;$$

$$x = 1 - l = 0.5;\ \ w_t = \left(\frac{1}{3}\right)(0.15)\left(\frac{2.3148}{0.5}\right)^{\frac{2}{3}} = 0.13889;$$

$$r = \rho + \delta_k = 0.06.$$

$$y_t = A_G\left(\frac{l_t}{k_t}\right)^{\gamma} k_t = 0.15\left(\frac{0.5}{2.314\,8}\right)^{\frac{1}{3}} 2.314\,8 = 0.208;$$

$$\begin{aligned} c_t &= A_G\left(\frac{l_t}{k_t}\right)^{\gamma} k_t - \delta_k k_t = 0.15\left(\frac{0.5}{2.315}\right)^{\frac{1}{3}} 2.315 - (0.03)\,2.315 \\ c_t &= 0.13889; \end{aligned}$$

$$i_t = \delta_k k_t = (0.03)\,2.3148 = 0.0694.$$

Productivity Change in Examples 8.2 and 9.2

- $A_G = 0.15 \longrightarrow A_G = (0.15)(1.05) = 0.1575.$

$$k = \frac{T\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)-\alpha\delta_k} = \frac{(1)\left(\frac{1}{3}\right)(0.1575)^3\left(\frac{2}{3(0.06)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.06)(1.5)-0.5(0.03)};$$
$$= 2.6797.$$

$$l_t = k_t\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right]^{\frac{1}{\gamma}} = k\left(\frac{(0.06)\,1.5}{A_G}\right)^3 = 2.6797\left(\frac{(0.06)\,1.5}{0.1575}\right)^3 = 0.5.$$

$$w_t = \gamma A_G\left(\frac{k}{l}\right)^{1-\gamma} = \left(\frac{1}{3}\right)(0.1575)\left(\frac{2.6797}{0.5}\right)^{\frac{2}{3}} = 0.16078.$$

Time Endowment Change: Examples 8.3, 9.3

$T = 1 \longrightarrow T = 1.05,$

$$k = \frac{(1.05)\,\gamma A_G^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \alpha\delta_k} = \frac{(1.05)\left(\frac{1}{3}\right)(0.15)\left(\frac{2(0.15)}{3(0.06)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.06)(1.5) - 0.5\,(0.03)};$$
$$= 2.4306.$$

$$l_t = 1.05 - \frac{\alpha k_t \left\{A_G\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right] - \delta_k\right\}}{\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}} = 0.525..$$

$$w_t = \gamma A_G \left(\frac{k}{l}\right)^{1-\gamma} = \left(\frac{1}{3}\right)(0.15)\left(\frac{2.4306}{0.525}\right)^{\frac{2}{3}} = 0.138\,89.$$

Business Cycle Explanation in Example 9.4

A 5% increase in A_G, T, and a 5% decrease in A_G, T :

$$
\begin{aligned}
k &= \frac{(1.05)\,\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)-\alpha\delta_k} = \frac{(1.05)\left(\frac{1}{3}\right)(0.1575)^3\left(\frac{2}{3(0.06)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.06)(1.5)-0.5\,(0.03)}; \\
&= 2.8137
\end{aligned}
$$

$$
\begin{aligned}
k &= \frac{(0.95)\,\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)-\alpha\delta_k} = \frac{(0.95)\left(\frac{1}{3}\right)(0.1425)^3\left(\frac{2}{3(0.06)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.06)(1.5)-0.5\,(0.03)}; \\
&= 1.8854.
\end{aligned}
$$

Depreciation Rate and Leisure Preference Change

- $\delta_k = 0.03$ to $\delta_k = 0.04$: k falls from 2.315 to 1.5117 :

$$k = \frac{T\gamma A_G \left[\frac{(1-\gamma)A_G}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)-\alpha\delta_k} = \frac{\left(\frac{1}{3}\right)(0.15)\left(\frac{2(0.15)}{3(0.07)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.07)(1.5)-0.5(0.04)}; \quad = 1.5117. \tag{6}$$

- $\alpha = 0.5 \longrightarrow \alpha = 0.6$: k rises from 2.315 to 2.1044 :

$$k = \frac{T\gamma A_G \left[\frac{(1-\gamma)A_G}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)-\alpha\delta_k} = \frac{\left(\frac{1}{3}\right)(0.15)\left(\frac{2(0.15)}{3(0.06)}\right)^2}{\left(\frac{1}{3}+0.6\right)(0.06)(1.5)-0.6(0.03)}; \quad = 2.1044 \tag{7}$$

Time Preference Change

- $\rho = 0.03$ to $\rho = 0.04$: $k = 2.15$ to $k = 1.41$.

$$k = \frac{T\gamma A_G \left[\frac{(1-\gamma)A_G}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \alpha\delta_k} = \frac{1\left(\frac{1}{3}\right)(0.15)\left(\frac{2(0.15)}{3(0.07)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.07)(1.5) - 0.5(0.03)}; = 1.4075 \qquad (8)$$

- Maximum capital stock (Unbounded utility): $\rho = 0$,

$$k = \frac{T\gamma A_G \left[\frac{(1-\gamma)A_G}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \alpha\delta_k} = \frac{1\left(\frac{1}{3}\right)(0.15)\left(\frac{2(0.15)}{3(0.03)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.03)(1.5) - 0.5(0.03)}; = 24.691. \qquad (9)$$

Labor Share Change

- $\gamma = \frac{1}{3}$ to $\gamma = 0.5$:

$$
\begin{aligned}
k &= \frac{T\gamma A_G \left[\frac{(1-\gamma)A_G}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \alpha\delta_k} = \frac{1\left(\frac{1}{2}\right)(0.15)\left(\frac{(0.15)}{2(0.06)}\right)}{\left(\frac{1}{2}+0.5\right)(0.06)(2) - 0.5(0.03)}; \\
&= 0.89286.
\end{aligned}
$$

$$
\begin{aligned}
l_t &= k_t \left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right]^{\frac{1}{\gamma}} = k\left(\frac{(0.06)\,2}{0.15}\right)^2 = k\,(0.64)\,, \\
l_t &= 0.89286\,(0.64) = 0.57143.
\end{aligned}
$$

- Capital to labor ratio falls by two-thirds from
  - $\frac{k_t}{l_t} = \frac{2.3148}{0.5} = 4.6296$ to $\frac{k_t}{l_t} = \frac{1}{0.64} = 1.5625.$

Unemployment, Fixed Wage, in Example 9.6

- $\bar{w} = 0.1389$, and recessionary 5% decrease in A_G and T.

$$k = \frac{T\gamma A_G\left[\frac{(1-\gamma)A_G}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{(\gamma+\alpha)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)-\alpha\delta_k} = \frac{0.95\left(\frac{1}{3}\right)(0.1425)\left(\frac{2(0.1425)}{3(0.06)}\right)^2}{\left(\frac{1}{3}+0.5\right)(0.06)(1.5)-0.5(0.03)};$$
$$= 1.8854.$$

$$l = \left(\frac{\gamma A_G}{\bar{w}}\right)^{\frac{1}{1-\gamma}} k = \left(\frac{0.1425}{3\,(0.13889)}\right)^{1.5} 1.8854 = 0.37708.$$
$$\frac{k}{l} = \frac{1.8854}{0.37708} = 5.0.$$

- Capital to labor ratio rises.

Labor Tax and Employment in General Equilibrium

$$x_t = T - l_t = \frac{\alpha c_t}{w_t\left(1-\tau_l\right)};\ c_t = y_t - i_t = A_G\left(\frac{l_t}{k_t}\right)^{\gamma} k_t - \delta_k k_t;$$

$$\rho + \delta_k = r_t = (1-\gamma) A_G \left(\frac{l_t}{k_t}\right)^{\gamma};\ \frac{l_t}{k_t} = \left[\frac{\rho+\delta_k}{(1-\gamma) A_G}\right]^{\frac{1}{\gamma}};$$

$$w_t = \gamma A_G \left(\frac{k}{l}\right)^{1-\gamma} = \gamma A_G^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}};$$

$$T - k_t\left[\frac{\rho+\delta_k}{(1-\gamma) A_G}\right]^{\frac{1}{\gamma}} = \frac{\alpha k_t\left(A_G\left[\frac{\rho+\delta_k}{(1-\gamma)A_G}\right] - \delta_k\right)}{(1-\tau_l)\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}};$$

$$k_t = k = \frac{T(1-\tau_l)\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{\left[(1-\tau_l)\gamma+\alpha\right]\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \alpha\delta_k}.$$

Labor Tax Equilibrium in Example 9.7

- $A_G = 0.15,\ T = 1,\ \gamma = 0.33,\ \alpha = 0.5,\ \delta_k = 0.03,\ \rho = 0.03,$
  - $\tau_l = 0.2:$

$$k_t = \frac{T(1-\tau_l)\gamma A_G^{\frac{1}{\gamma}}\left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{\left[(1-\tau_l)\gamma+\alpha\right]\left(\frac{\rho+\delta_k}{(1-\gamma)}\right)-\alpha\delta_k}$$

$$= \frac{(1-0.2)\frac{1}{3}(0.15)^3\left(\frac{2}{3(0.03+0.03)}\right)^2}{\left(\frac{(1-0.2)}{3}+0.5\right)\left(\frac{3(0.03+0.03)}{2}\right)-0.5(0.03)} = 2.0576.$$

  - Capital stock decreases 11.1% from 2.3148.
- $\tau_l = 0.3;\ k_t = 1.9063.$

Application: Capital Derived Graphically

- Contributed by Michal Kejak.
- Four-sectioned, or quadrant, diagram:
  - northeast, southeast, southwest, northwest.
- Northeast (NE): $AS - AD$ plus consumption demand.
- Southeast (SE): Marginal rate of substitution, goods-leisure.
- Southwest (SW): An identity: $k = (k/l)\,l$.
- Northwest (NW): Capital to labor ratio.

Northeast: AS-AD plus Consumption Market.

- $\left(\frac{1}{w}, c\right)$ spacial dimensions

$$AD : \frac{1}{w_t} = \frac{1}{y_t^d (1+\alpha) - [\rho + \delta(1+\alpha)] k_t},$$

$$AS : \frac{1}{w_t} = \frac{\left(y_t^s\right)^{\frac{1-\gamma}{\gamma}}}{\gamma A_G^{\frac{1}{\gamma}} k_t^{\frac{1-\gamma}{\gamma}}}.$$

$$\frac{1}{w_t} = \frac{1}{c_t^d (1+\alpha) - \rho k_t}.$$

$$c_t^s = y_t - \rho k_t.$$

Southeast: MRS between Goods and Leisure.

- (c, l) spacial dimensions

$$\frac{x}{\alpha c} = \frac{1}{w},\ x_t = 1 - l_t,$$

$$l = 1 - \frac{\alpha c}{w}.$$

Southwest: An identity:

$k = (k/h)\, h.$

- $\left(l, \frac{k}{l}\right)$ spacial dimensions

$$l_t = \frac{k_t}{\left(\frac{k_t}{l_t}\right)}. \qquad (10)$$

- Links labor l_t of Southeast to capital stock k_t of Northwest
- allows k_t to be identified in graph,
  - on Southwest/Northwest horizontal axis.

Northwest: Firm's Capital to Labor Ratio.

- $\left(\frac{k}{l}, \frac{1}{w}\right)$ spacial dimensions.

$$\frac{w_t}{r_t} = \frac{\gamma}{1-\gamma}\frac{k_t}{l_t}.$$

$$r_t = \rho + \delta_k$$

$$\begin{aligned} w_t &= r_t \frac{\gamma}{1-\gamma}\frac{k_t}{l_t} = (\rho + \delta_k)\frac{\gamma}{1-\gamma}\frac{k_t}{l_t}; \\ \frac{1}{w} &= \frac{1-\gamma}{\gamma}\frac{1}{\frac{k_t}{l_t}(\rho + \delta)}. \end{aligned}$$

- A hyperbola with $\frac{k_t}{l_t}$ in denominator.

Example Quadrant Graph: Baseline Dynamic Model

- Northeast equations:

$$\frac{1}{w_t} = \frac{1}{y_t^d\left(1+0.5\right)-2.3148\left[0.03+\left(1.5\right)0.03\right]}, \tag{11}$$

$$\frac{1}{w_t} = \frac{3\left(y_t^s\right)^2}{\left(0.15\right)^3 k_t^2}, \tag{12}$$

$$\frac{1}{w_t} = \frac{1}{c_t^d\left(1+0.5\right)-\left(0.03\right)k_t}, \tag{13}$$

$$c_t^s = y_t-\left(0.03\right)k_t. \tag{14}$$

Northeast Graph

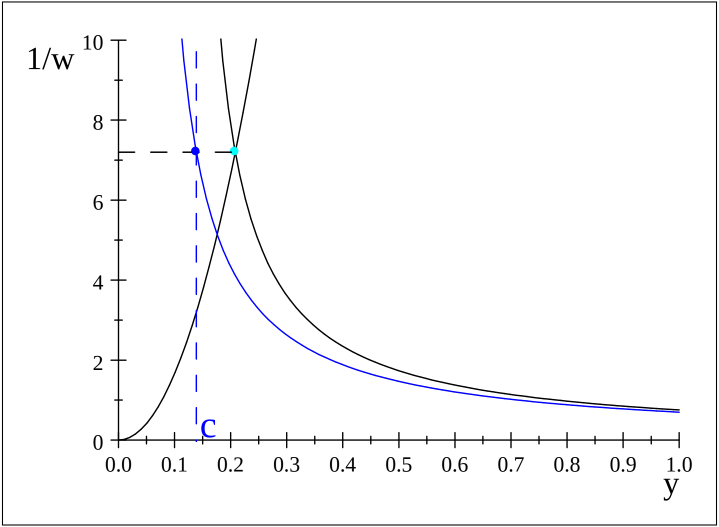

Figure 10.1. Northeast Quadrant $AS - AD$ with Consumption Demand and Supply in Example 8.1.

Normalized Northeast Equations: Divide by 10

- $\left(\frac{1}{w(10)}, y\right)$ spacial dimensions

$$\frac{1}{w_t(10)} = \left(\frac{1}{10}\right)\frac{1}{y_t^d(1+0.5) - 2.3148\left[0.03 + (1.5)\,0.03\right]}, \quad (15)$$

$$\frac{1}{w_t(10)} = \left(\frac{1}{10}\right)\frac{3\left(y_t^s\right)^2}{(0.15)^3\,k_t^2}, \quad (16)$$

$$\frac{1}{w_t(10)} = \left(\frac{1}{10}\right)\frac{1}{c_t^d(1+0.5) - (0.03)\,k_t}, \quad (17)$$

$$c_t^s = y_t - (0.03)\,k_t. \quad (18)$$

Normalized Northeast Graph

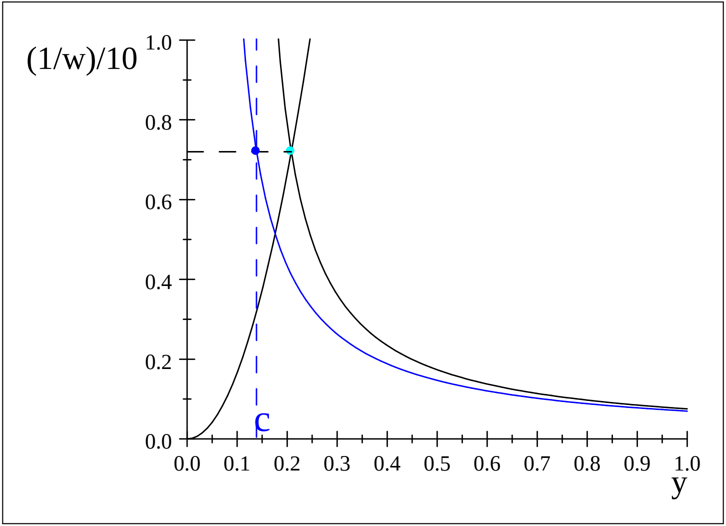

Figure 10.2. Northeast Quadrant $AS - AD$ Normalized with $\left(\frac{1}{w_t}\right) / (10)$, in Example 8.1.

Southeast Equations

- $$l_t = 1 - \left(\frac{\alpha}{w_t}\right) c_t = 1 - \left(\frac{0.5}{0.13889}\right) c_t, \quad (19)$$
  $$= 1 - (3.6)\, c_t. \quad (20)$$
- Slope of line: $-\left(\frac{\alpha}{w_t}\right) = -3.6$;
- equilibrium labor is $l_t = 0.5$.
- At $l_t = 0$, $c_t = 0.278$,

$$0 = l_t = 1 - \left(\frac{\alpha}{w_t}\right) c_t,$$

$$c_t = \frac{w_t}{\alpha} = \frac{0.13889}{0.5} = 0.27778. \quad (21)$$

  - value of c_t on the horizontal axis where line intersects.

Southeast Graph

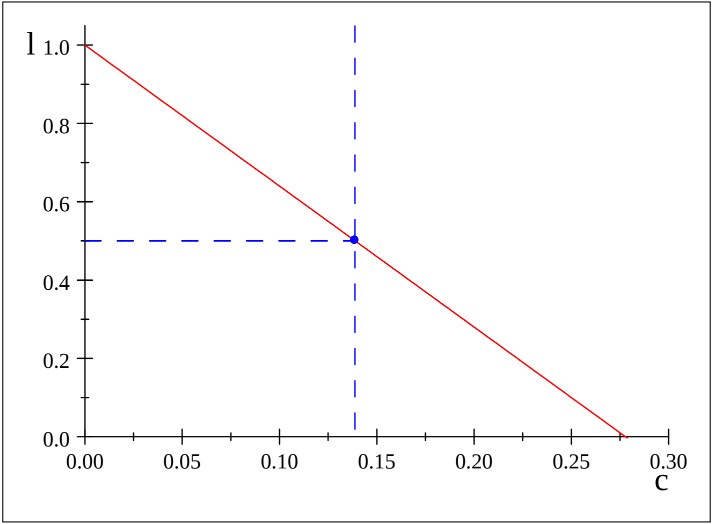

Figure 10.3. Southeast Quadrant: Consumer Goods and Labor in Example 8.1.

Southwest Equations

- $$l_t = \frac{k_t}{\left(\frac{k_t}{l_t}\right)} = \frac{2.3148}{\left(\frac{k_t}{l_t}\right)} \tag{22}$$
- Graph is hyperbola, with $k_t = 2.3148$.
- Dotted blue lines: equilibrium $l = 0.5$ on vertical axis,
  - and equilibrium $\frac{k_t}{l_t}$ on horizontal axis,

$$\frac{k_t}{l_t} = \frac{2.3148}{0.5} = 4.6296,$$

- Black dashed lines: where $l_t = 1.0$ on vertical axis,
  - and $k_t = 2.31\,48$ on horizontal axis:

$$\frac{k_t}{l_t} = \frac{2.3148}{1.0} = 2.3148.$$

  as shown on the horizontal axis.
  - This is equilibrium value of k_t graphically,
  - the primary goal of graphs: to identify k_t.

Southwest Graph

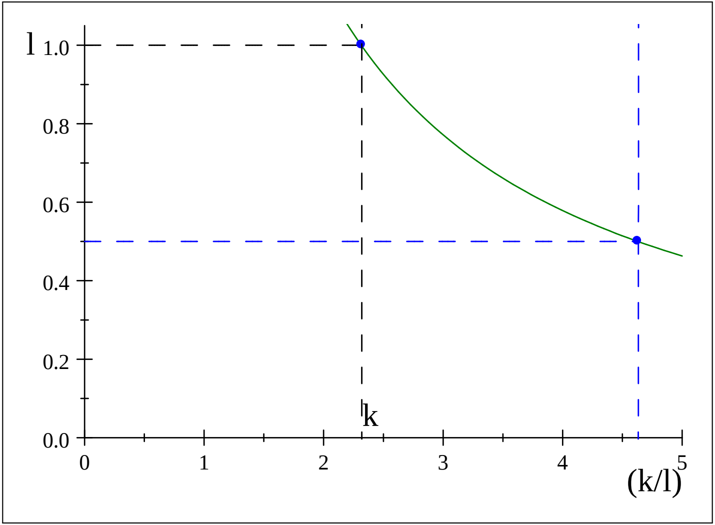

Figure 10.4. Southwest Quadrant: Labor, the Capital to Labor Ratio, and with the Capital Stock Identified, in Example 8.1.

Southwest Normalized Equations

- $\left(l : \frac{k_t}{l_t(10)}\right)$ spacial dimensions.
- Everywhere k_t divided by 10:

$$l_t = \frac{\left(\frac{2.3148}{10}\right)}{\left(\frac{k_t}{l_t(10)}\right)}, \tag{23}$$

- Dotted blue lines: $l = 0.5$, on vertical axis,
  - and on horizontal axis:

$$\frac{k_t}{l_t\,(10)} = \frac{2.3148}{0.5\,(10)} = 0.46296.$$

- Dashed Black lines indicate $l_t = 1.0$ on vertical axis,
  - and $\frac{k_t}{(10)} = 0.23418$ on horizontal axis.

$$\frac{k_t}{l_t\,(10)} = \frac{2.3148}{1.0\,(10)} = 0.23148,$$

  - to identify normalized k_t, which is $\frac{k_t}{(10)}$.

Southwest Normalized Graph

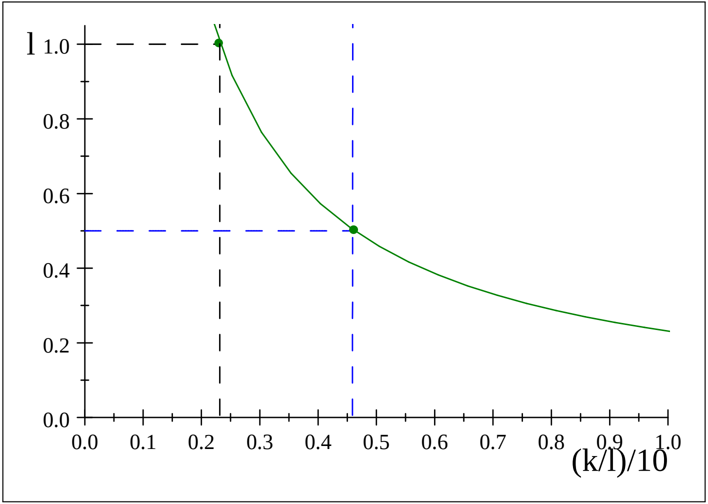

Figure 10.5. Southwest Quadrant: Labor and Normalized Capital to Labor Ratio $\frac{k_t}{l_t(10)}$ of Example 8.1.

Northwest Equations

$$\frac{1}{w} = \frac{\frac{1-\gamma}{\gamma}\frac{1}{\rho+\delta}}{\frac{k_t}{l_t}} = \frac{2}{(0.06)\,\frac{k_t}{l_t}}.$$

- $\left(\frac{1}{w_t} : \frac{k_t}{l_t(10)}\right)$ spacial dimensions for normalization.

$$\frac{1}{w_t} = \frac{2}{(0.06)\,(10)\left(\frac{k_t}{l_t(10)}\right)} = \frac{2}{(0.60)\left(\frac{k_t}{l_t(10)}\right)}. \tag{24}$$

- $\frac{k_t}{l_t(10)}$ enters denominator: gives a hyperbola.
- Blue dashed vertical line: $\frac{k_t}{l_t(10)} = \frac{0.23148}{0.5} = 0.46296.$
- Black black dashed line: $\frac{1}{w_t} = 7.2,$
- which links back to the black dashed line
  - in $AS - AD$ Northeast graph in first quadrant.

Northwest Graph

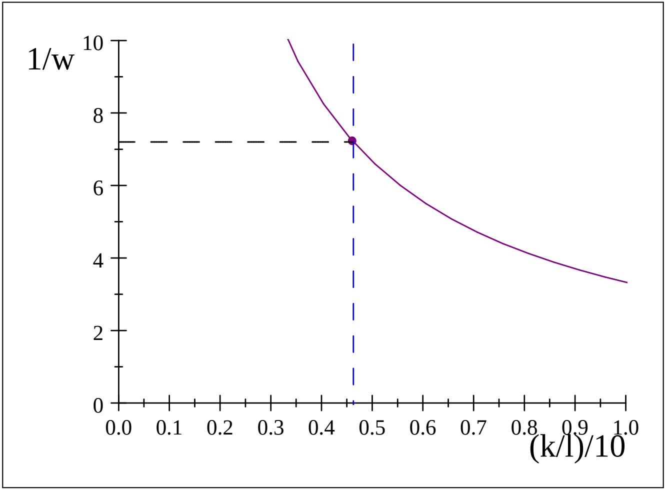

Figure 10.6. Northwest Quadrant: Firm's Equilibrium Relation between Relative Price and Normalized Capital to Labor Ratio in Example 8.1.

Northwest Further Normalization

- Also dividing $\frac{1}{w_t}$ by 10,
- $\left(\frac{1}{w_t(10)}, \frac{k_t}{l_t(10)}\right)$ spacial dimensions.

$$\frac{1}{w_t\,(10)} = \left(\frac{1}{10}\right)\frac{\frac{1-\gamma}{\gamma}\frac{1}{\rho+\delta}}{\frac{k_t}{l_t}} \tag{25}$$

$$= \left(\frac{1}{10}\right)\frac{2}{(0.06)\,(10)\left(\frac{k_t}{l_t(10)}\right)} \tag{26}$$

$$= \frac{2}{(0.60)\left(\frac{k_t}{l_t(10)}\right)}. \tag{27}$$

Northwest Graph Further Normalized

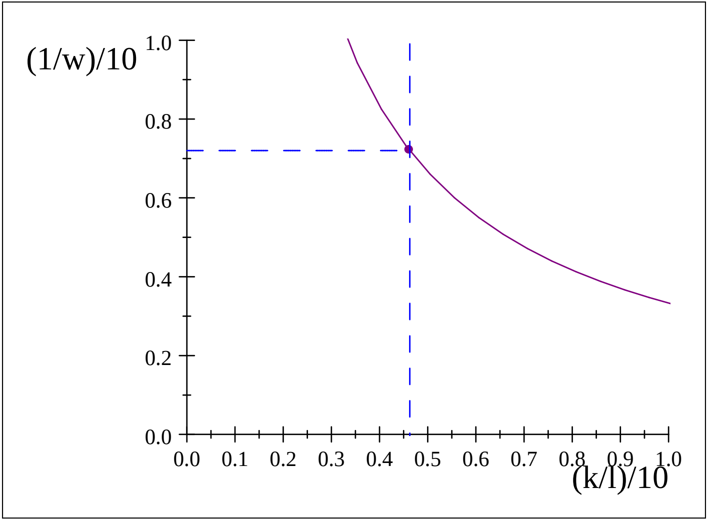

Figure 10.7. Northwest Quadrant: Normalized in Example 8.1.

Four Quadrant Graph

- Capital stock indicated
- by intersection of vertical black dashed line
- with $\frac{k_t}{l_t(10)}$ horizontal axis,
- indicated by light green circle-dot on horizontal axis
- where $\frac{k_t}{l_t(10)} = 0.23148$.

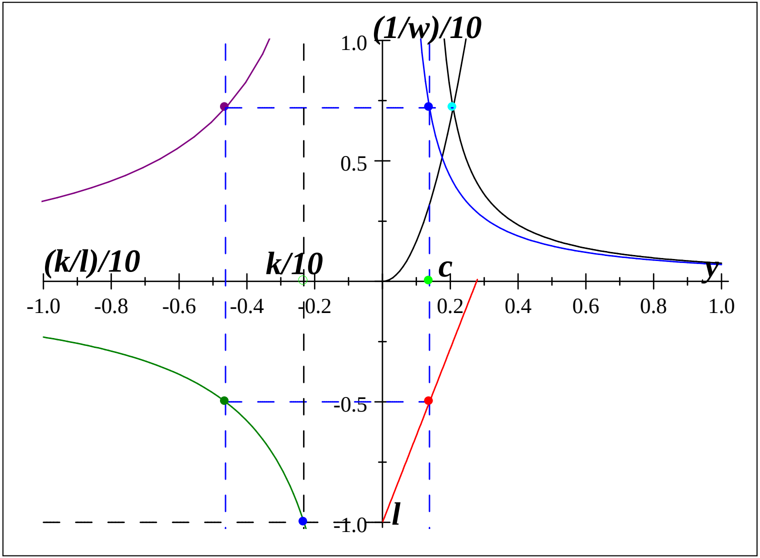

Figure 10.8. Four Quadrant Equilibrium in Example 8.1, with k_t Determined at Black Vertical Dashed Line.

Example 8.2 Productivity Increase

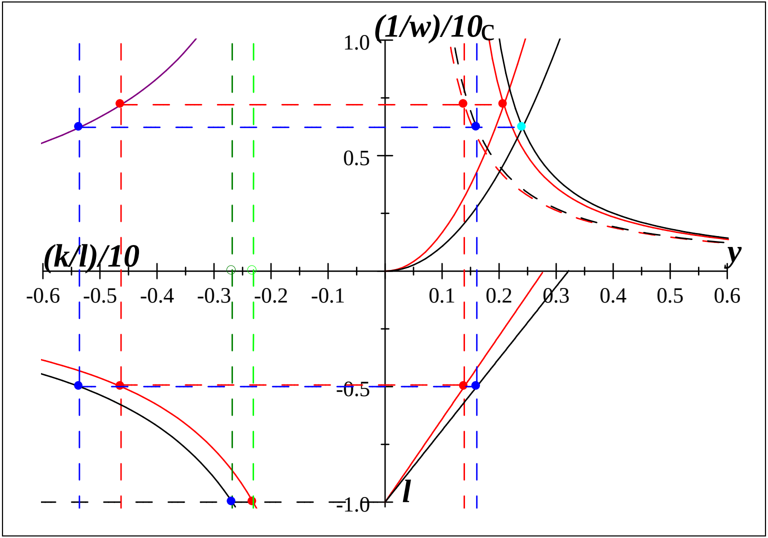

Figure 10.9. Comparative Static of an Increase in Goods Productivity A_G in Example 8.2 Compared to Baseline.

Example 8.2 Productivity Increase

- Shift from red to black lines.
- Capital stock rises to 2.6797