Advanced Modern Macroeconomics

Exogenous Growth

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- Exogenous technological progress in goods sector
  - Output growth rate a simple function of productivity growth.
  - Account for basic facts about the growth process.
- Modification to consumption demand and $AS - AD$.
- Interest rate no longer simple sum
  - of time preference rate and depreciation rate;
  - interest rate still function of exogenous parameters
  - because of exogeneous growth rate.
- Calibration two percent targeted growth rate.
  - Same interest rate and wage rate as before
  - if time preference rate is lowered.
- Goods, Labor growth shown over 4 periods.
- Trend downwards in time endowment
  - captures trend upwards in education time.
  - and trend downwards in labor hours per week.

Building on the Last Chapters

- Parts 2, 3, 4 allow technology progress
  - in terms of increase in output productivity parameter.
  - Dynamic Part 4 assumes zero exogenous growth.
- Now exogenous growth rate is positive instead of zero
  - through continuous increase in output productivity.
- Calibration is shown with same interest rate
  - by having lower rate of time preference.
- Goods productivity factor increases at constant rate over time;
  - standard neoclassical growth model.
- Adds time endowment decrease at small rate over time.
- Two comparative statics explain trends in growth facts, employment.
  - Still consistent with fluctuations around trends for business cycle.
  - Growth theory with trends in parameters consistent
  - with business cycle theory of changes in parameters.

Learning Objective

- Baseline dynamic model can have positive exogenous growth.
- Focus on intertemporal marginal rate of substitution.
- How interest rate is determined once growth rate is given.
- Then revise $AS - AD$ framework.
- Graphical changes of $AS - AD$ as in growth facts.
- Use same comparative static tools:
  - goods and time endowment changes,
  - as used for business cycles.

Who Made It Happen

- Solow 1956 extended Ramsey model of growth.
  - by introducing technological change.
  - simultaneously done by Swan 1956 in Australia.
- Subsequent articles by Solow on "growth accounting"
  - compute contribution of labor, capital, technological change.
  - 1957 "Technical Change & Aggregate Production Function"
  - 1960 "Investment and Technical Progress".
- Ramsey's 1928 set out dynamic optimization model
  - Solow worked on firm side with constant savings rate.
  - R.G.D.Allen's 1968 *Macro-Economic Theory*:
  - focus on growth but with no utility maximization.
- 1965 David Cass, T.C Koopmans independently
  - combined Solow growth with Ramsey optimization:
  - Ramsey-Cass-Koopmans neoclassical growth model.
  - Brock and Mirman 1972 included uncertainty
  - with random shock in productivity, as in RBC.

Causes of Sustained Growth

- Shift up in output productivity parameter causes sustained growth.
- Principle of sustained growth:
  - each factor of production must grow continuously.
  - Physical capital factor, k_t, can already grow:

$$k_{t+1} = k_t \left(1 - \delta_k\right) + i_t.$$

  - If $i_t \geq \delta_k k_t$, then capital is growing.
- Need labor to grow as well, or "labor augmentation".
- Productivity shift acts as labor augmentation.
- Raw labor time cannot grow indefinitely:
  - limited amount of time endowment.
  - Growth models in essence must augment labor time
- *Total Factor Productivity* (*TFP*) growth: common approach.

Exogenous Technological Progress

- *TFP* increase same as A_G increase
- Let A_G depend on time: denoted A_{Gt}, where

$$y_t = A_{Gt} \left(l_t \right)^{\gamma} \left(k_t \right)^{1-\gamma}. \tag{1}$$

- Assume A_{Gt} grows at rate μ :

$$A_{Gt+1} = A_{Gt} \left(1 + \mu \right), \tag{2}$$

- Re-write production function with A_{Gt} factoring labor time:

$$y_t = \left[l_t \left(A_{Gt} \right)^{\frac{1}{\gamma}} \right]^{\gamma} \left(k_t \right)^{1-\gamma}. \tag{3}$$

- Define $\tilde{A}_{Gt} \equiv \left(A_{Gt} \right)^{\frac{1}{\gamma}}$, then

$$y_t = \left[l_t \tilde{A}_{Gt} \right]^{\gamma} \left(k_t \right)^{1-\gamma}.$$

- Result: if both factors $l_t \tilde{A}_{Gt}$ and k_t grow at same rate
  - then output y_t grows at that same rate.

Stylized Growth Facts

- Real wage rises over time
- Real interest rate remains constant.
- Output to capital ratio remains constant
- Per capita income rises.

Growth Facts by Assuming Trend in TFP

$$\frac{y_t}{k_t} = \left[\frac{l_t \tilde{A}_{Gt}}{k_t}\right]^{\gamma},$$

$$r_t = MP_{k_t} = (1-\gamma)\frac{y_t}{k_t} = (1-\gamma)\left[\frac{l_t \tilde{A}_{Gt}}{k_t}\right]^{\gamma}$$

$$\Longrightarrow \Delta\left(\frac{l_t \tilde{A}_{Gt}}{k_t}\right) = 0;\ \Delta(l_t) = 0, \Longrightarrow \Delta\left(\tilde{A}_{Gt}\right) = \Delta k_t$$

$$1+g = \frac{y_{t+1}}{y_t} = \frac{\left[l_{t+1}\left(A_{Gt+1}\right)^{\frac{1}{\gamma}}\right]^{\gamma}\left(k_{t+1}\right)^{1-\gamma}}{\left[l_t\left(A_{Gt}\right)^{\frac{1}{\gamma}}\right]^{\gamma}\left(k_t\right)^{1-\gamma}} = (1+\mu)\left(\frac{k_{t+1}}{k_t}\right)^{1-\gamma},$$

$$\frac{k_{t+1}}{k_t} = \left(\frac{1+g}{1+\mu}\right)^{\frac{1}{1-\gamma}};\ \left(\frac{A_{Gt+1}}{A_{Gt}}\right)^{\frac{1}{\gamma}} = (1+\mu)^{\frac{1}{\gamma}}.$$

$$\Longrightarrow \left(\frac{1+g}{1+\mu}\right)^{\frac{1}{1-\gamma}} = (1+\mu)^{\frac{1}{\gamma}};\ 1+g = (1+\mu)^{\frac{1}{\gamma}}$$

Growth Accounting

- Baseline calibration: $\gamma = \frac{1}{3}$.
  - also assume growth rate of $g = 0.03$,

$$\implies \quad g = 0.03 = 3\mu = \frac{\mu}{\gamma},$$
$$\implies \quad \mu = 0.01.$$

- $$g = \mu + (1-\gamma)\frac{k_{t+1}}{k_t} = \mu + (1-\gamma)\frac{\mu}{\gamma}.$$
  - capital accounts for $(1-\gamma)\frac{\mu}{\gamma} = 2\mu = 0.02$,
  - two-thirds of total $g = 0.03$.
  - Technology change: one-third of growth
- Can be reinterpreted with endogenous growth, human capital.

Balanced Growth Path Equilibrium

- All variables that grow do so at common rate g.
- y_t, c_t, k_t and i_t grow at rate g, for any t :

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{i_{t+1}}{i_t} = 1 + g.$$

- Time allocations x_t, l_t stationary along BGP : i.e. constant.
- Consumer Demand along BGP : $\frac{k_{t+1}}{k_t} = 1 + g$,

$$\begin{aligned} c_t^d &= Tw_t l_t + r_t k_t - k_{t+1} + k_t (1 - \delta_k), \\ &= Tw_t l_t + r_t k_t - k_t (1 + g) + k_t (1 - \delta_k), \\ &= Tw_t l_t + k_t (r_t - \delta_k - g). \end{aligned}$$

$$\begin{aligned} l_t &= 1 - \frac{\alpha c_t}{w_t}, \\ c_t^d &= \frac{Tw_t + k_t (r_t - \delta_k - g)}{1 + \alpha}. \end{aligned}$$

Intertemporal Marginal Rate of Substitution along BGP

$$
\begin{aligned}
1+g &= \frac{\left(c_{t+1}\right)^{d}}{\left(c_{t}\right)^{d}}=\frac{1+r_{t}-\delta_{k}}{1+\rho}, \\
r_{t}-\delta_{k} &= (1+g)(1+\rho)-1, \\
r_{t}-\delta_{k}-g &= (1+g)(1+\rho)-1-g=\rho(1+g).
\end{aligned}
$$

$$
\begin{aligned}
c_{t}^{d} &= \frac{1}{1+\alpha}\left[T w_{t}+k_{t} \rho(1+g)\right], \\
y_{p t} &\equiv T w_{t}+k_{t} \rho(1+g), \\
c_{t}^{d} &= \frac{y_{p t}}{1+\alpha}.
\end{aligned}
$$

Same as $c_t^d = \frac{1}{1+\alpha}\left(Tw_t + k_t\rho\right)$ when $g = 0$ in Part 4.

AS-AD with Growth

$$y_t^d = c_t^d + s_t;\ s_t = k_{t+1}^s - k_t^s\left(1-\delta_k\right),\ k_t^d = k_t^s = k_t;$$

$$s_t = k_t^s\left[\frac{k_{t+1}^s}{k_t^s} - \left(1-\delta_k\right)\right] = k_t^s\left[\left(1+g\right) - \left(1-\delta_k\right)\right] = k_t^s\left(g+\delta_k\right).$$

$$y_t^d = \frac{Tw_t + k_t\rho\left(1+g\right)}{1+\alpha} + k_t\left(g+\delta_k\right),$$

$$AD : \frac{1}{w_t} = \frac{T}{y_t^d\left(1+\alpha\right) - k_t\left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left(1+\alpha\right)\right]}$$

$$y_t^s = A_G\left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}}\left(k_t\right)^{\gamma}\left(k_t\right)^{1-\gamma} = A_G^{\frac{1}{1-\gamma}}\left(\frac{\gamma}{w_t}\right)^{\frac{\gamma}{1-\gamma}}k_t,$$

$$AS : \frac{1}{w_t} = \frac{\left(y_t^s\right)^{\frac{1-\gamma}{\gamma}}}{\gamma A_G^{\frac{1}{\gamma}}\left(k_t\right)^{\frac{1-\gamma}{\gamma}}}.$$

$\gamma = \frac{1}{3}$, $\alpha = 0.5$, $\rho = 0.03$, $T = 1$, $\delta_k = 0.03$, $A_{Gt} = 0.15$;
$g = 0.02$; $.k_t = 1.0161$.

$$\begin{aligned} 1.02 &= 1 + g = (1+\mu)^{\frac{1}{\gamma}} = (1+\mu)^3, \\ \mu &= (1.02)^{\frac{1}{3}} - 1 = 0.0066227. \end{aligned}$$

$$\begin{aligned} \frac{1}{w_t} &= \frac{T}{y_t^d (1+\alpha) - k_t \left[\rho(1+g) + (g+\delta_k)(1+\alpha)\right]} \\ \frac{1}{w_t} &= \frac{1}{y_t^d (1.5) - k_t \left[0.03(1.02) + (0.02+0.03)(1.5)\right]}, \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{1}{w_t} &= \frac{\left(y_t^s\right)^{\frac{1-\gamma}{\gamma}}}{\gamma A_{Gt}^{\frac{1}{\gamma}} (k_t)^{\frac{1-\gamma}{\gamma}}} \\ \frac{1}{w_t} &= \frac{3\left(y_t^s\right)^2}{(0.15)^3 (1.0161)^2}. \end{aligned} \tag{5}$$

Shift Back in AS-AD with 2% Growth

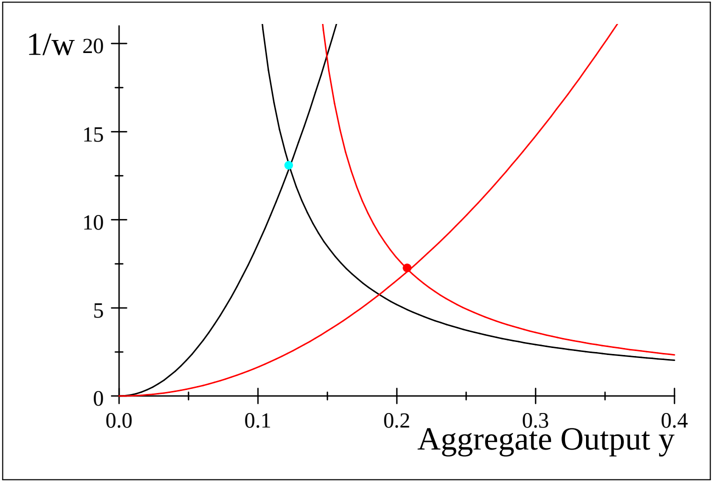

Figure 11.1. $AS - AD$ Equilibrium With 2% Exogenous Growth and Zero Growth in Example 11.1.

Recalibration: Example 11.2

With $r_t = (1+g)(1+\rho) + \delta_k - 1$,
then $g > 0$ implies r_t decreases; k_t decreases.
With $r = 0.06$ and $g = 0.02$, then change ρ :

$$\begin{aligned}
r_t &= (1+g)(1+\rho) + \delta_k - 1, \\
r_t &= (1+0.02)(1+\rho) + 0.03 - 1 = 0.06; \\
\rho &= \frac{1.03}{(1.02)} - 1 = 0.0098; \implies k_t = 2.7778.
\end{aligned}$$

$$\begin{aligned}
\frac{1}{w_t} &= \frac{1}{y_t^d (1.5) - 2.7778 \left[0.0098039 (1.02) + (0.02 + 0.03)(1.5)\right]}, \\
\frac{1}{w_t} &= \frac{3 \left(y_t^s\right)^2}{(0.15)^3 (2.7778)^2}.
\end{aligned}$$

AS-AD Shift Out, Wage the Same

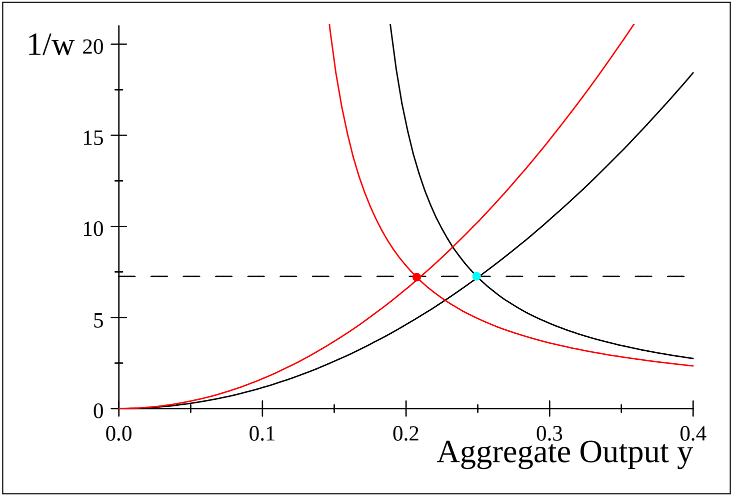

Figure 11.2. $AS - AD$ Equilibrium With 2% Exogenous Growth and a Lower Rate of Time Discount

Consumption and Output

$$
\begin{aligned}
c_t^d &= \frac{1}{1+\alpha}\left[Tw_t + k_t\rho\left(1+g\right)\right] \\
&= \frac{2}{3}\left(0.13889 + (2.7778)(0.0098)(1.02)\right) = 0.1111.
\end{aligned}
$$

$$
\begin{aligned}
y_t^d &= \frac{1}{1+\alpha}\left(Tw_t + k_t\left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left(1+\alpha\right)\right]\right) \\
&= \frac{2}{3}\left(0.1389 + (2.7778)\left((0.0098)(1.02) + (0.05)(1.5)\right)\right) = 0.25.
\end{aligned}
$$

$$
\left(\delta_k + g\right)k_t = (0.03 + 0.02)(2.7778) = 0.13889.
$$

- c falls by 20%, k, y rise by 20%, i rises by 100%.
- Consumption to output ratio falls from two-thirds to

$$
\frac{c^d}{y^d} = \frac{0.1111}{0.25} = 0.444.
$$

Labor Market

$$
\begin{aligned}
T - l_t^s &= x_t = \frac{\alpha c_t}{w_t};\ c_t^d = \frac{1}{1+\alpha}\left[T w_t + k_t \rho\left(1+g\right)\right], \\
l_t^s &= T - \frac{\alpha}{1+\alpha}\left[T + \frac{k_t \rho\left(1+g\right)}{w_t}\right]; \\
l_t^d &= \left(\frac{\gamma A_G}{w_t}\right)^{\frac{1}{1-\gamma}} k_t.
\end{aligned}
$$

$$
\begin{aligned}
w_t &= \frac{\alpha k_t \rho\left(1+g\right)}{T - \left(1+\alpha\right) l_t^s} = \frac{\left(0.5\right)\left(2.7778\right)\left(0.0098\right)\left(1.02\right)}{1 - \left(1.5\right) l_t^s}, \\
w_t &= \gamma A_G \left(\frac{k_t}{l_t}\right)^{1-\gamma} = \frac{1}{3}\left(0.15\right)\left(\frac{\left(2.7778\right)}{l_t^d}\right)^{\frac{2}{3}}.
\end{aligned}
$$

Shift out in Labor Supply and Demand

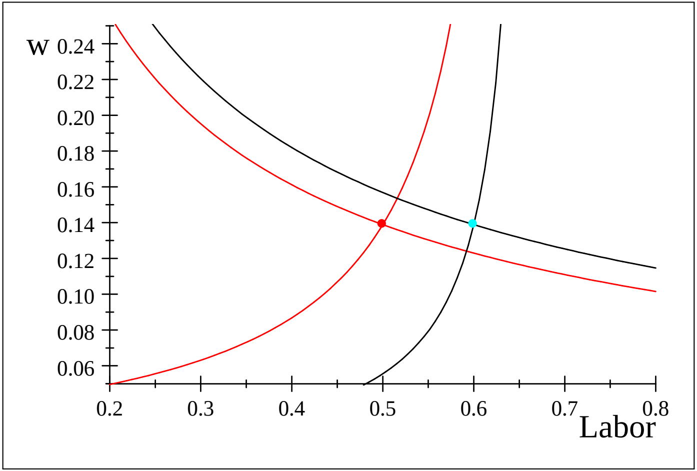

Figure 11.3. Higher Employment, Same Wage, $g = 0.02$ of Example 11.2

The Exogenous Growth Process Over Time

Example 11.3. Continuous Shifts in Productivity

$g = 0.02$, $\mu = 0.00662$.

$$
\begin{aligned}
A_{Gt} &= 0.15,\ A_{Gt+1} = 0.15\left(1+\mu\right). \\
A_{Gt+1} &= (0.15)\,(1.0066227) = 0.151, \\
A_{Gt+2} &= (0.150\,99)\,(1.0066227) = 0.152, \\
A_{Gt+3} &= (0.15199)\,(1.0066227) = 0.153.
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \quad k_t = 2.7778,\ k_{t+1} = 2.8331, \\
k_{t+2} &= 2.89,\ k_{t+3} = 2.9478.
\end{aligned}
$$

AS-AD Over Time

$$\frac{1}{w_{t+1}} = \frac{1}{y^d_{t+1}(1.5) - 2.8331\left[(0.0098039)(1.02) + (0.05)(1.5)\right]},$$

$$\frac{1}{w_{t+1}} = \frac{3\left(y^s_{t+1}\right)^2}{(0.15099)^3(2.8331)^2};$$

$$\frac{1}{w_{t+2}} = \frac{1}{y^d_{t+2}(1.5) - 2.8898\left[(0.0098039)(1.02) + (0.05)(1.5)\right]},$$

$$\frac{1}{w_{t+2}} = \frac{3\left(y^s_{t+2}\right)^2}{(0.15199)^3(2.89)^2};$$

$$\frac{1}{w_{t+3}} = \frac{1}{y^d_{t+3}(1.5) - 2.9478\left[(0.0098039)(1.02) + (0.05)(1.5)\right]},$$

$$\frac{1}{w_{t+3}} = \frac{3\left(y^s_{t+3}\right)^2}{(0.153)^3(2.9478)^2}.$$

Output and Real Wage Rises

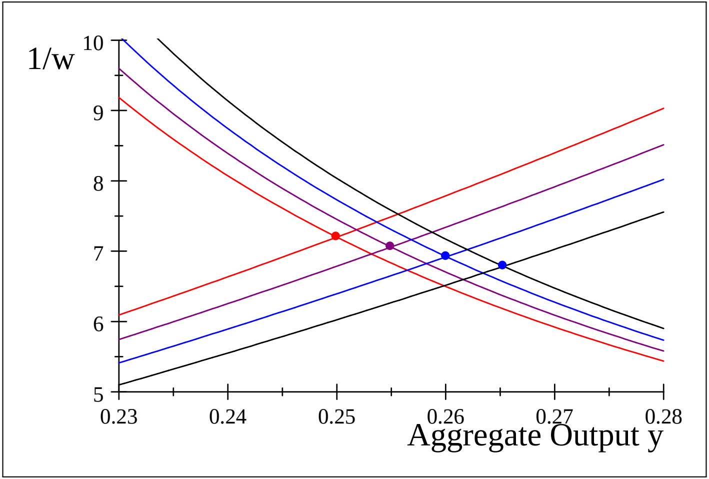

Figure 11.4. $AS - AD$ Equilibria Over Time With 2% Exogenous Growth in Example 11.3.

Labor Market

$$
\begin{aligned}
w_{t+1} &= \frac{(0.5)(2.8331)(0.0098039)(1.02)}{1-(1.5)\, l^s_{t+1}}, \\
w_{t+1} &= \frac{1}{3}(0.15099)\left((2.8331)\,/\left(l^d_{t+1}\right)\right)^{\frac{2}{3}};
\end{aligned}
$$

$$
\begin{aligned}
w_{t+2} &= \frac{(0.5)(2.8898)(0.0098039)(1.02)}{1-(1.5)\, l^s_{t+2}}, \\
w_{t+2} &= \frac{1}{3}(0.15199)\left((2.89)\,/\left(l^d_{t+2}\right)\right)^{\frac{2}{3}};
\end{aligned}
$$

$$
\begin{aligned}
w_{t+3} &= \frac{(0.5)(2.9478)(0.0098039)(1.02)}{1-(1.5)\, l^s_{t+3}}, \\
w_{t+3} &= \frac{1}{3}(0.153)\left((2.9478)\,/\left(l^d_{t+3}\right)\right)^{\frac{2}{3}}.
\end{aligned}
$$

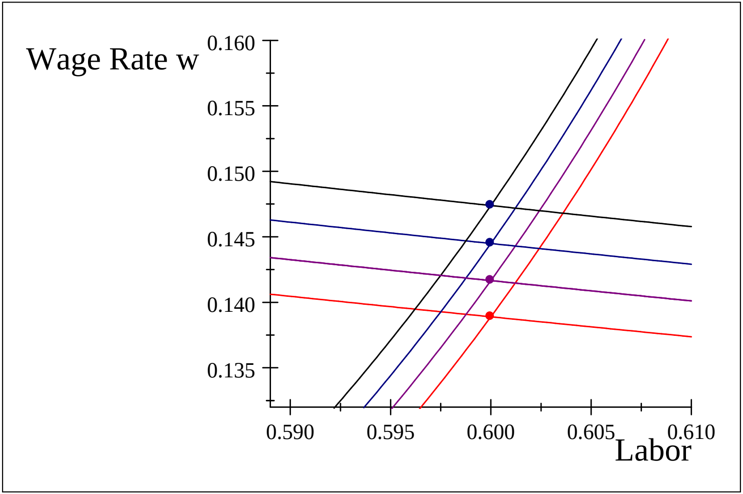

Figure 11.5. Labor Market with 2% Exogenous Growth in Example 11.3.

Consumption and Output

Consumption and Output grow at 2% rate:

$$
\begin{aligned}
c_{t+1}^{d} &= \frac{2}{3}\left(0.14166+(2.8331)(0.0098)(1.02)\right)=0.11333;\\
c_{t+2}^{d} &= \frac{2}{3}\left(0.144\,49+(2.89)(0.0098)(1.02)\right)=0.11559;\\
c_{t+3}^{d} &= \frac{2}{3}\left(0.14739+(2.9478)(0.0098)(1.02)\right)=0.11791.
\end{aligned}
$$

$$
\begin{aligned}
y_{t+1}^{d} &= \frac{2}{3}\left(0.142+(2.83)\left((0.0098)(1.02)+(0.05)(1.5)\right)\right)=0.255,\\
y_{t+2}^{d} &= \frac{2}{3}\left(0.144+(2.89)\left((0.0098)(1.02)+(0.05)(1.5)\right)\right)=0.260,\\
y_{t+3}^{d} &= \frac{2}{3}\left(0.147+(2.95)\left((0.0098)(1.02)+(0.05)(1.5)\right)\right)=0.265.
\end{aligned}
$$

Growth in Input Market Space

Isocost Lines Over Time

$$
\begin{aligned}
ISOCOST &: \quad y_t = w_t l_t + r_t k_t, \\
0.25 &= (0.13889)\, l_t + (0.06)\, k_t, \\
k_t &= \frac{0.25}{0.06} - \frac{(0.13889)\, l_t}{0.06}; \\
k_{t+1} &= \frac{0.25498}{0.06} - \frac{(0.14166)}{0.06} l_{t+1}; \\
k_{t+2} &= \frac{0.26008}{0.06} - \frac{(0.14449)}{0.06} l_{t+2}; \\
k_{t+3} &= \frac{0.2653}{0.06} - \frac{(0.14739)}{0.06} l_{t+3}
\end{aligned}
$$

$$ISOQUANT \quad : \quad 0.25 = y_t = A_G \left(l_t^d\right)^{\gamma} (k_t)^{1-\gamma} = 0.15 \left(l_t^d\right)^{\frac{1}{3}\gamma} (k_t)^{\frac{2}{3}},$$

$$k_{t+1} = \frac{\left(\frac{(0.25498)}{0.15099}\right)^{\frac{3}{2}}}{\left(l_{t+1}^d\right)^{\frac{1}{2}}};$$

$$k_{t+2} = \frac{\left(\frac{(0.26008)}{0.15199}\right)^{\frac{3}{2}}}{\left(l_{t+2}^d\right)^{\frac{1}{2}}};$$

$$k_{t+2} = \frac{\left(\frac{(0.2653)}{0.153}\right)^{\frac{3}{2}}}{\left(l_{t+2}^d\right)^{\frac{1}{2}}}.$$

Growth in Input Market Space

Input Ratios Over Time

$$\frac{k_t}{l_t} = \frac{2.7778}{0.60} = 4.6297;$$

$$\frac{k_{t+1}}{l_{t+1}} = \frac{2.8331}{0.60} = 4.7218;$$

$$\frac{k_{t+2}}{l_{t+2}} = \frac{2.89}{0.60} = 4.816;$$

$$\frac{k_{t+3}}{l_{t+3}} = \frac{2.9478}{0.60} = 4.913.$$

Isocosts, Isoquants Shift up, Labor Unchanged

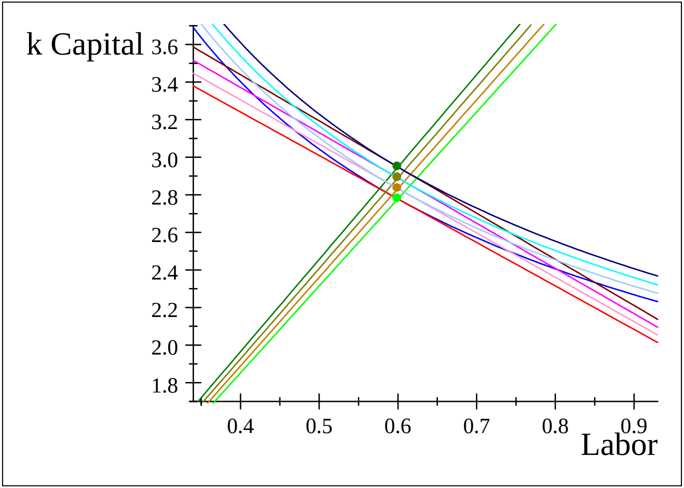

Figure 11.6. Factor Market Equilibrium with 2% Exogenous Growth in Example 11.3.

Growth in Output Space

Production Possibility Curves Over Time

$$
\begin{aligned}
c_t^d &= y_t^s - i_t = A_G \left(l_t^d \right)^{\gamma} (k_t)^{1-\gamma} - (g + \delta_k)\, k_t, \\
c_t^d &= (0.15) \left(l_t^d \right)^{\frac{1}{3}} (2.7778)^{\frac{2}{3}} - (0.02 + 0.03)\,(2.7778)\,; \\
c_{t+1}^d &= (0.15099) \left(l_t^d \right)^{\frac{1}{3}} (2.8331)^{\frac{2}{3}} - (0.02 + 0.03)\,(2.8331)\,; \\
c_{t+2}^d &= (0.15199) \left(l_t^d \right)^{\frac{1}{3}} (2.89)^{\frac{2}{3}} - (0.02 + 0.03)\,(2.89)\,; \\
c_{t+3}^d &= (0.153) \left(l_t^d \right)^{\frac{1}{3}} (2.9478)^{\frac{2}{3}} - (0.02 + 0.03)\,(2.9478)\,;
\end{aligned}
$$

$$u = \ln c_t + \alpha \ln x_t = \ln c_t + \alpha \ln (1 - l_t),$$

$$-2.4527 = \ln 0.1111 + 0.5 \ln 0.4,$$

$$-2.4527 = \ln c_t + 0.5 \ln (1 - l_t),$$

$$c_t = \frac{e^{\ln 0.1111 + 0.5 \ln 0.4}}{(1 - l_t)^{0.5}}; \quad (6)$$

$$c_{t+1} = \frac{e^{\ln 0.11333 + 0.5 \ln 0.4}}{(1 - l_{t+1})^{0.5}}; \quad (7)$$

$$c_{t+2} = \frac{e^{\ln 0.11559 + 0.5 \ln 0.4}}{(1 - l_{t+2})^{0.5}}; \quad (8)$$

$$c_{t+3} = \frac{e^{\ln 0.11791 + 0.5 \ln 0.4}}{(1 - l_{t+3})^{0.5}}. \quad (9)$$

Growth in Output Space

Budget Lines Over Time

$$\begin{aligned}
c_t^d &= w_t l_t^s + \rho \left(1+g\right) k_t^s, \\
c_t^d &= \left(0.13889\right) l_t^s + \left(0.0098039\right)\left(1+0.02\right)\left(2.7778\right); \\
c_{t+1}^d &= \left(0.14166\right) l_{t+1}^s + \left(0.0098039\right)\left(1+0.02\right)\left(2.8331\right); \\
c_{t+2}^d &= \left(0.14449\right) l_{t+2}^s + \left(0.0098039\right)\left(1+0.02\right)\left(2.8898\right); \\
c_{t+3}^d &= \left(0.14739\right) l_{t+3}^s + \left(0.0098039\right)\left(1+0.02\right)\left(2.9478\right).
\end{aligned}$$

Production, Utility Shift up, Labor Unchanged

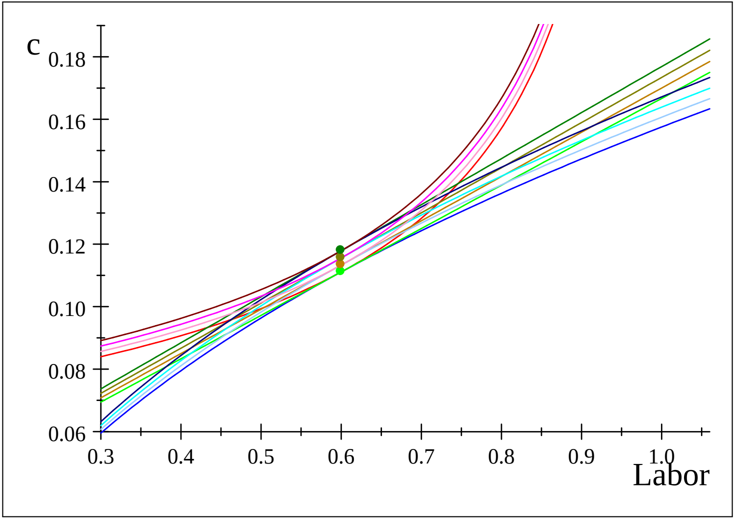

Figure 11.7. General Equilibrium Consumption and Utility Levels with Exogenous Growth

Trend in Time Endowment

- Aguiar, Hurst 2009: US 1965-2005
  - labor hours down 12%, leisure up 5%
  - $T = x + l$, net decrease of 7% over 40 years
  - T down by 0.00182 or 0.182% per year
  - over 40 years, since $(1 - 0.00182)^{40} = 0.93$
  - 7% decrease from 1.
- New Experiment: Let T trend down, A_G trend up
  - changing in opposite directions simultaneously,
  - related to business cycle explanation:
  - Chapters 3, 9 : T and A_G increased in expansion.
  - Now T trends down slightly by 0.182%
  - while A_G trends up more significantly.by 0.72%.

Example 11.4: Opposite Trends in Time, Productivity

$$\begin{aligned} A_{Gt} &= 0.15,\ \ T_t = 1; \\ A_{Gt+1} &= 0.15108,\ \ T_{t+1} = 1 - 0.00182 = 0.99818; \\ A_{Gt+2} &= 0.15216,\ \ T_{t+2} = (0.99818)(1 - 0.00182) = 0.9964; \\ A_{Gt+3} &= 0.15327,\ \ T_{t+3} = (0.99636)(1 - 0.00182) = 0.9946. \end{aligned}$$

$$k_t = 2.7778, k_{t+1} = 2.8280, k_{t+2} = 2.8793., k_{t+3} = 2.9317,$$

$$g = 0.02.$$

Example 11.4: AS-AD

$$\frac{1}{w_{t+1}} = \frac{(0.99818)}{y^d_{t+1}(1.5) - (2.8331)\left[(0.0098)(1.02) + (0.05)(1.5)\right]},$$

$$\frac{1}{w_{t+1}} = \frac{3\left(y^s_{t+1}\right)^2}{(0.15108)^3(2.8331)^2};$$

$$\frac{1}{w_{t+2}} = \frac{(0.99636)}{y^d_{t+2}(1.5) - (2.89)\left[(0.0098)(1.02) + (0.05)(1.5)\right]},$$

$$\frac{1}{w_{t+2}} = \frac{3\left(y^s_{t+2}\right)^2}{(0.15216)^3(2.89)^2};$$

$$\frac{1}{w_{t+3}} = \frac{(0.99455)}{y^d_{t+3}(1.5) - (2.948)\left[(0.0098)(1.02) + (0.05)(1.5)\right]},$$

$$\frac{1}{w_{t+3}} = \frac{3\left(y^s_{t+3}\right)^2}{(0.15327)^3(2.948)^2}.$$

AS-AD Shift Out, Wage Rises Over Time

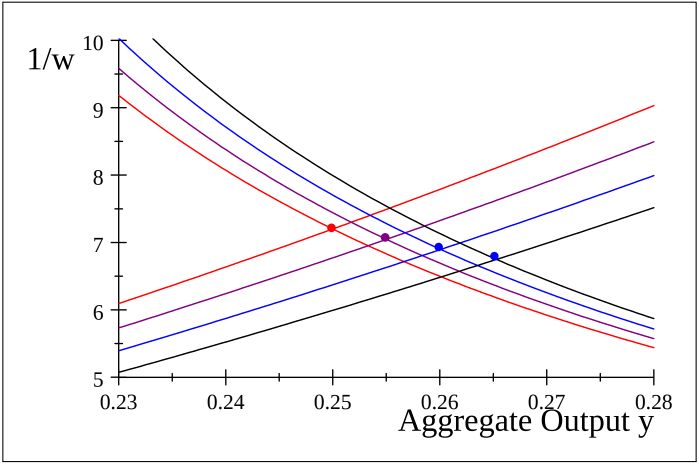

Figure 11.8. $AS - AD$ Equilibria With A_G Trending Up and T Trending Down in Example 11.4.

Labor Market Supply, Demand Over Time

$$
\begin{aligned}
w_{t+1} &= \frac{(0.5)(2.8337)(0.0098039)(1.02)}{(0.99818)-(1.5)\,l^{s}_{t+1}}, \\
w_{t+1} &= \frac{1}{3}(0.15108)\left[(2.8331)\,/\,\left(l^{d}_{t+1}\right)\right]^{\frac{2}{3}};
\end{aligned}
$$

$$
\begin{aligned}
w_{t+2} &= \frac{(0.5)(2.89)(0.0098039)(1.02)}{(0.99636)-(1.5)\,l^{s}_{t+2}}, \\
w_{t+2} &= \frac{1}{3}(0.15216)\left[(2.8902)\,/\,\left(l^{d}_{t+2}\right)\right]^{\frac{2}{3}};
\end{aligned}
$$

$$
\begin{aligned}
w_{t+3} &= \frac{(0.5)(2.948)(0.0098039)(1.02)}{(0.99455)-(1.5)\,l^{s}_{t+3}}, \\
w_{t+3} &= \frac{1}{3}(0.15327)\left[(2.948)\,/\,\left(l^{d}_{t+3}\right)\right]^{\frac{2}{3}}.
\end{aligned}
$$

Labor Supply Shifts Back with Hours Falling

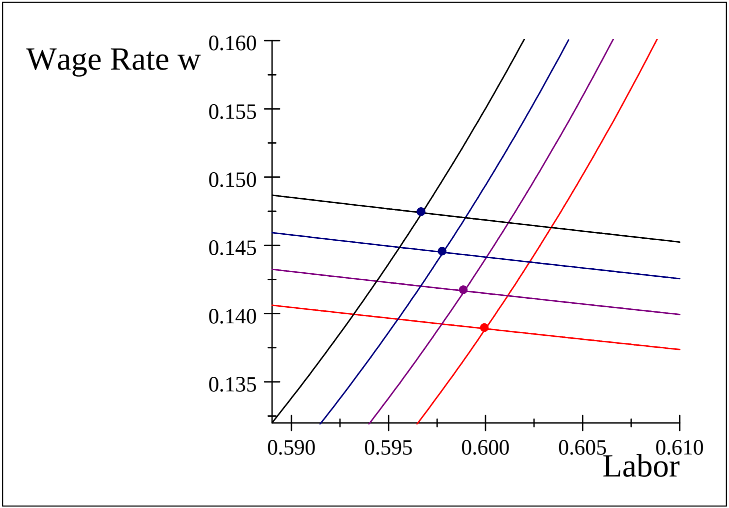

Figure 11.9. Labor Market with A_G Trending Up and T Trending Down in Example 11.4.

Employment Falls Slightly Over Time

$$
\begin{aligned}
l_t &= 0.6, \\
l_{t+1} &= (0.6)(1 - 0.00182) = 0.598\,91, \\
l_{t+2} &= (0.598\,91)(1 - 0.00182) = 0.597\,82, \\
l_{t+3} &= (0.597\,82)(1 - 0.00182) = 0.596\,73.
\end{aligned}
$$

TFP, Japan's Lost Decade, Minnesota School

- A_{Gt} :"total factor productivity", or *TFP*, the "Solow residual".
- Much research computing *TFP*, including over time
- Hayashi and Prescott 2002: Japan's 1990s stagnation
  - "lost decade" explained by a low *TFP*.
  - *TFP* accounting used both for growth, business cycles
  - given the cycle is more of a longer term,
  - like occasional change in the trend growth rate.
- Kehoe, Prescott 2002 "Great Depressions of the Twentieth Century",
  - known as "Minnesota approach", including
  - 1930s depression in Germany, France, Italy, US and UK,
  - 1980s contractions in Argentina, Mexico, Chile
  - 1990s Japan .
- Extended in Chari et al 2007 "Business Cycle Accounting".

Appendix A.11: Solution Methodology for Exogenous Growth

$$
\begin{aligned}
1+g &= \frac{c_{t+1}}{c_t} = \frac{1+r_t-\delta_k}{1+\rho}, \\
&\Longrightarrow g\left(1+\rho\right)+\rho+\delta_k = r_t = \left(1-\gamma\right)A_G\left(\frac{l_t}{k_t}\right)^{\gamma}, \\
&\Longrightarrow \frac{l_t}{k_t} = \left[\frac{g\left(1+\rho\right)+\rho+\delta_k}{\left(1-\gamma\right)A_G}\right]^{\frac{1}{\gamma}}; w_t = \gamma A_G\left(\frac{l_t}{k_t}\right)^{\gamma-1}
\end{aligned}
$$

$$
y_t^d = \frac{w_t T + k_t\left[\rho\left(1+g\right)+\left(1+\alpha\right)\left(g+\delta_k\right)\right]}{1+\alpha} = A_G\left(\frac{l_t}{k_t}\right)^{\gamma} k_t = y_t^s.
$$

Solving for the Capital Stock from AS-AD

- Substitute into $y_t^d = y_t^s$ equation for w as a function of $\frac{l_t}{k_t}$
  - from marginal product of labor,
- and substitute in for $\frac{l_t}{k_t}$,
  - solved from intertemporal margin
  - and the marginal product of capital.

$$\Longrightarrow k_t =$$

$$\frac{T\gamma\left(A_G\right)^{\frac{1}{\gamma}}\left(\frac{(1-\gamma)}{g(1+\rho)+\rho+\delta_k}\right)^{\frac{1-\gamma}{\gamma}}}{(1+\alpha)\,\delta_k\left(\frac{\gamma}{(1-\gamma)}\right)+(1+\alpha)\left(\frac{g(1+\rho)+\rho}{(1-\gamma)}\right)-\left[\rho\left(1+g\right)+\left(1+\alpha\right)g\right]}$$

Example 11.1: Capital Stock Solution

$\gamma = \frac{1}{3}$, $\alpha = 0.5$, $\rho = 0.03$, $T = 1$, $\delta_k = 0.03$, $A_{Gt} = 0.15$, $g = 0.02$.

$$\Longrightarrow k_t = 1.0161 =$$

$$\frac{(0.15)^3 \left(\frac{2}{3(0.02(1.03)+0.03+0.03)} \right)^2}{3\left((1.5)(0.03)(0.5) + 1.5\left(\frac{3(0.02(1.03)+0.03)}{2} \right) - (0.03(1.02) + (1.5)(0.02)) \right.}$$

$g = 0.04$ instead of $g = 0.02$, then the capital stock falls almost by half to 0.53357 :

$$g = 0.04 \Longrightarrow k_t = 0.53357 =$$

$$\frac{(0.15)^3 \left(\frac{2}{3(0.04(1.03)+0.03+0.03)} \right)^2}{3\left((1.5)(0.03)(0.5) + 1.5\left(\frac{3(0.04(1.03)+0.03)}{2} \right) - (0.03(1.04) + (1.5)(0.04)) \right.}$$

Examples 11.2: Capital Stock Solution

$r = 0.06$, $g = 0.02$, $\rho = 0.0098039$

$$\Longrightarrow k_t = 2.7778 =$$

$$\frac{(0.15)^3 \left(\frac{2}{3(0.02(1.0098)+0.0098+0.03)} \right)^2}{3\left((1.5)\,(0.03)\,(0.5) + 1.5\left(\frac{3(0.02(1.0098)+0.0098)}{2} \right) - (0.0098\,(1.02) + 0.03) \right)}$$

Examples 11.3: Continuous Productivity Increase

$$\implies k_{t+1} = 2.8331 =$$

$$\frac{(0.15099)^3 \left(\frac{2}{3(0.02(1.0098039)+0.0098+0.03)}\right)^2}{3\left((1.5)(0.03)(0.5)+1.5\left(\frac{3(0.02(1.0098)+0.0098)}{2}\right)-(0.0098(1.02)+0.03)\right)}.$$

$$\implies k_{t+2} = 2.89 =$$

$$\frac{(0.15199)^3 \left(\frac{2}{3(0.02(1.0098)+0.0098+0.03)}\right)^2}{3\left((1.5)(0.03)(0.5)+1.5\left(\frac{3(0.02(1.0098)+0.0098)}{2}\right)-(0.0098(1.02)+0.03)\right)}$$

$$\implies k_{t+3} = 2.9478 =$$

$$\frac{(0.153)^3 \left(\frac{2}{3(0.02(1.0098)+0.0098+0.03)}\right)^2}{3\left((1.5)(0.03)(0.5)+1.5\left(\frac{3(0.02(1.0098)+0.0098)}{2}\right)-(0.0098(1.02)+0.03)\right)}.$$

Examples 11.4: Trends in Time, Goods

$$\Longrightarrow k_{t+1} = 2.8337 =$$

$$\frac{0.99818\left(0.15108\right)^{3}\left(\frac{2}{3(0.02(1.0098)+0.0098+0.03)}\right)^{2}}{3\left(\left(1.5\right)\left(0.03\right)\left(0.5\right)+1.5\left(\frac{3(0.02(1.0098)+0.0098)}{2}\right)-\left(0.0098\left(1.02\right)+0.03\right)\right)}$$

$$\Longrightarrow k_{t+2} = 2.8902 =$$

$$\frac{0.99636\left(0.15216\right)^{3}\left(\frac{2}{3(0.02(1.0098)+0.0098+0.03)}\right)^{2}}{3\left(\left(1.5\right)\left(0.03\right)\left(0.5\right)+1.5\left(\frac{3(0.02(1.0098)+0.0098)}{2}\right)-\left(0.0098\left(1.02\right)+0.03\right)\right)}$$

$$\Longrightarrow k_{t+3} = 2.9480 =$$

$$\frac{0.99455\left(0.15327\right)^{3}\left(\frac{2}{3(0.02(1.0098)+0.0098+0.03)}\right)^{2}}{3\left(\left(1.5\right)\left(0.03\right)\left(0.5\right)+1.5\left(\frac{3(0.02(1.0098)+0.0098)}{2}\right)-\left(0.0098\left(1.02\right)+0.03\right)\right)}$$

Examples 11.4: Growth Rate

$$\frac{k_{t+1}}{k_t} = \frac{\dfrac{T_{t+1}\gamma(A_{Gt+1})^{\frac{1}{\gamma}}\left(\frac{(1-\gamma)}{g(1+\rho)+\rho+\delta_k}\right)^{\frac{1-\gamma}{\gamma}}}{(1+\alpha)\delta_k\left(\frac{\gamma}{(1-\gamma)}\right)+(1+\alpha)\left(\frac{g(1+\rho)+\rho}{(1-\gamma)}\right)-[\rho(1+g)+(1+\alpha)g]}}{\dfrac{T_t\gamma(A_{Gt})^{\frac{1}{\gamma}}\left(\frac{(1-\gamma)}{g(1+\rho)+\rho+\delta_k}\right)^{\frac{1-\gamma}{\gamma}}}{(1+\alpha)\delta_k\left(\frac{\gamma}{(1-\gamma)}\right)+(1+\alpha)\left(\frac{g(1+\rho)+\rho}{(1-\gamma)}\right)-[\rho(1+g)+(1+\alpha)g]}},$$

$$\frac{k_{t+1}}{k_t} = \frac{T_{t+1}\left(A_{Gt+1}\right)^{\frac{1}{\gamma}}}{T_t\left(A_{Gt}\right)^{\frac{1}{\gamma}}} = \frac{T_t\left(1-0.00182\right)\left[A_{Gt}\left(1.0072\right)\right]^{\frac{1}{\gamma}}}{T_t\left(A_{Gt}\right)^{\frac{1}{\gamma}}}$$

$$= \left(1-0.00182\right)\left(1.0072\right)^3 = 1.02.$$