Advanced Modern Macroeconomics

Human Capital and Endogenous Growth

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- Extends dynamic exogenous growth
  - human capital investment sector,
  - leading to endogenous growth.
- Human capital enters goods production,
  - with its growth comparable to productivity growth.
  - Instead of goods productivity rising exogenously.
- Separate new sector, adds 2nd intertemporal margin
  - through this growth rate endogenous.
  - *BGP* returns on human, physical capital equal.
  - State variable: ratio of physical to human capital.
- $AS - AD$ presented with modifications:
  - account for time in human capital production,
  - human capital productivity parameter.sector.
- Special cases: only human capital, only physical capital.
- Growth facts over 4 periods yet no parameter change.

Solution Methodology

- $AS - AD$ used to get single equation in one unknown.
- Complication: a quadratic equation, two roots.
  - Rule out economically infeasible solution.
- Solution methodology in Appendix $A12$.
- Here, solve quadratic for g, growth rate
  - instead of capital stock,
  - since calibration targets a growth rate value.

Building on the Last Chapters

- Directly extend exogenous growth model of Part 4.
  - Human capital made alternative growth process.
  - Part 4 fixed interest rate becomes endogenous.
- Shows Part 4 $AS - AD$ analysis be extended
  - leave main structure of $AS - AD$ framework,
  - while showing growth process without assumption
  - of exogenously changing parameters.over time.
- Rather than chapter on solution methodology
  - as in Chapter 10,
  - methodology in chapter appendix.

Learning Objective

- Challenge: to see few modifications to $AS - AD$.
- Requiring complication of solving quadratic.
- Understanding: endogenizing BGP growth rate g
  - also endogenizes interest rate.
- Visualizing growth process over time,
  - without changing any parameters,
  - also a primary objective of chapter.

Who Made It Happen

- Human capital part of labor theory
  - in Adam Smith; Alfred Marshall:
  - "Most valuable of all capital is that invested in human beings"
  - Theodore W. Schultz: Nobel Prize citing human capital.
  - Jacob Mincer, 1974 book "Schooling, Experience, Earnings".
  - Gary Becker: Formalization of theory.
- New growth theory of 1960s
  - Hirofumi Uzawa 1962, 1963 two-sector growth models
  - still using Solow assumption of fixed consumption rate.
  - 1965: technological change function of education time.
- Lucas 1967 costs of adjusting physical capital stock,
  - like Uzawa (1962,1963), but dynamic utility maximizing.
  - Extended with Prescott 1971 with Solow growth.
- Lucas 1988: cost of adjusting human capital with growth,
  - following Uzawa 1965 costly human capital adjustment.
  - Combines Uzawa 1965 with Ramsey framework.

Human Capital Investment

- Human capital, h_t, with $h_t = 1$, and $w_t h_t$ effective wage rate;
  - $l_t h_t$ "effective labor time".
  - Time augmented by human capital.
  - w_t wage of "raw" labor stable along BGP
  - instead of rising as when exogenous growth.
  - Per capita income rises only because human capital rises.
- Goods production function

$$y_t^s = A_G \left(l_t^d h_t \right)^{\gamma} (k_t)^{1-\gamma}.$$

  - $l_t^d h_t$ and capital k_t rising at rate g; so is output.

Relation to Exogenous Growth

- $$\begin{aligned} y_t^s &= \left(l_t^d h_t\right)^{\gamma} (k_t)^{1-\gamma}, \\ h_t &= (A_{Gt})^{\frac{1}{\gamma}}, \\ y_t^s &= A_{Gt} \left(l_t^d\right)^{\gamma} (k_t)^{1-\gamma}. \end{aligned}$$
    - A_{Gt} growing instead of human capital: exogenous growth.
    - Exogenous human capital growth:
    - would simply rename exogenous growth productivity factor A_{Gt}.
- Second Sector and Endogenous Growth
    - accumulation of capital over time given by

$$\begin{aligned} h_{t+1} &= h_t (1-\delta_h) + i_{ht}, \\ k_{t+1} &= k_t (1-\delta_k) + i_t. \end{aligned}$$

    - $\delta_h \in [0, 1]$ depreciation rate, i_{ht} amount invested in human capital.

Investment in Human Capital

- Assume labor only linear production of human capital;
  - simplification (Lucas 1988); l_{Ht}: human capital time

$$i_{ht} = A_H l_{Ht} h_t.$$

- Allocation of time constraint

$$1 = l_{Ht} + l_t^s + x_t.$$

  - Exogenous growth comparison: define T_t :

$$\begin{aligned} T_t &\equiv l_t^s + x_t = 1 - l_{Ht}, \\ 1 &= T_t + l_{Ht}. \end{aligned}$$

  - T_t changes endogenously, in constrast to Parts $3, 4$.
- Central trade-off:
  - extra time in human capital means less work, leisure,
  - but higher stream of future earnings.

Growth with Human and Physical Capital

$$c_t^d = w_t l_t^s h_t + r_t k_t - k_{t+1} + k_t - \delta_k k_t.$$

- Two state variables, k_t, h_t :

$$\begin{aligned} V(k_t, h_t) &= \underset{k_{t+1}, h_{t\_1}, l_t^s, l_{Ht}, x_t}{Max} : \ln\left[w_t l_t^s h_t + r_t k_t - k_{t+1} + k_t - \delta_k k_t\right] \\ &+ + \alpha \ln(x_t) \beta V(k_{t+1}, h_{t+1}), \end{aligned}$$

$$\begin{aligned} h_{t+1} &= h_t(1 - \delta_h) + A_H l_{Ht} h_t, \\ x_t &= 1 - l_{Ht} - l_t^s. \end{aligned}$$

$$\begin{aligned} V(k_t, h_t) &= \underset{k_{t+1}, l_t^s, l_{Ht},}{Max} : \ln\left[w_t l_t^s h_t + r_t k_t - k_{t+1} + k_t - \delta_k k_t\right] \\ &+ + \alpha \ln(1 - l_{Ht} - l_t^s) \beta V\left[k_{t+1}, h_t(1 - \delta_h) + A_H l_{Ht} h_t\right] \end{aligned}$$

Equilibrium Conditions with Human Capital

- 3 first-order, 2 envelope conditions, for each k_t, h_t.

$$
\begin{aligned}
k_{t+1} &: \quad \frac{1}{c_t^d}(-1) + \beta \frac{\partial V(k_{t+1}, h_{t+1})}{\partial k_{t+1}} = 0, \\
l_t^s &: \quad \frac{1}{c_t^d}(w_t h_t) + \frac{\alpha}{x_t}(-1) = 0, \\
l_{Ht} &: \quad \frac{\alpha}{x_t}(-1) + \beta \frac{\partial V(k_{t+1}, h_{t+1})}{\partial h_{t+1}}(A_H h_t) = 0;
\end{aligned}
$$

$$
\begin{aligned}
k_t &: \quad \frac{\partial V(k_t, h_t)}{\partial k_t} = \frac{1}{c_t^d}(1 + r_t - \delta_k), \\
h_t &: \quad \frac{\partial V(k_t, h_t)}{\partial h_t} = \frac{1}{c_t^d}(w_t l_t) + \beta \frac{\partial V(k_{t+1}, h_{t+1})}{\partial h_{t+1}}(1 + A_H l_{Ht} - \delta_H).
\end{aligned}
$$

$$
1 + g_t = \frac{1 + r_t - \delta_k}{1 + \rho}; \; MRS_{c,x} : \frac{\alpha c_t^d}{x_t} = w_t h_t.
$$

Plus Second Intertemporal Capital Margin

- Take h_{t+1} first-order condition, bring back one time period:

$$
\begin{aligned}
\frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial h_{t+1}} &= \frac{\alpha}{\beta\left(A_H h_t\right) x_t};\ \frac{\partial V\left(k_t, h_t\right)}{\partial h_t} = \frac{\alpha}{\beta\left(A_H h_{t-1}\right) x_{t-1}}, \\
\frac{\partial V\left(k_t, h_t\right)}{\partial h_t} &= \frac{1}{c_t^d}\left(w_t l_t\right) + \beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial h_{t+1}}\left(1 + A_H l_{Ht} - \delta_H\right); \\
\frac{\left(h_t\right) x_t}{\left(h_{t-1}\right) x_{t-1}} &= \beta\left(A_H\right) l_t^s + \beta\left(1 + A_H l_{Ht} - \delta_H\right).
\end{aligned}
$$

- $BGP: \frac{x_t}{x_{t-1}} = 1;\ \frac{h_t}{h_{t-1}} = 1 + g$; and $l_t^s + l_{Ht} = 1 - x_t \ \left(\beta \equiv \frac{1}{1+\rho}\right)$:

$$
\begin{aligned}
1 + g_t &= \frac{1 + A_H\left(1 - x_t\right) - \delta_h}{1 + \rho} = \frac{1 + r_t - \delta_k}{1 + \rho}; \\
&\Longrightarrow \quad r_t - \delta_k = A_H\left(1 - x_t\right) - \delta_h.
\end{aligned}
$$

Growth Now Depends on Several Factors

- Growth rate g depends on
  - A_H human capital investment productivity
  - time productively employed, $1 - x$,
  - marginal product of physical capital r_t.
- Links directly: employment rate and growth rate.

AS-AD with Human Capital

- State variable is now $\frac{k_t}{h_t}$ rather than k_t;
- output is normalized by human capital: $\frac{y_t}{h_t}$.

$$\begin{aligned}
c_t^d &= w_t l_t^s h_t + r_t k_t - \left[k_t\left(1+g\right) - k_t\left(1-\delta_k\right)\right] \\
&= w_t l_t^s h_t + k_t\left(r_t - \delta_k - g\right). \\
\frac{c_t \alpha}{w_t h_t} &= x_t,\ x_t = 1 - l_t^s - l_{Ht};\ \Rightarrow l_t^s = 1 - \frac{c_t^d \alpha}{w_t h_t} - l_{Ht}. \\
c_t^d &= w_t h_t \left(1 - \frac{c_t^d \alpha}{w_t h_t} - l_{Ht}\right) + k_t\left(r_t - \delta_k - g\right), \\
c_t^d &= \frac{1}{1+\alpha}\left[w_t h_t\left(1 - l_{Ht}\right) + k_t\left(r_t - \delta_k - g\right)\right]; \\
1+g &= \frac{1+r-\delta_k}{1+\rho};\ \Rightarrow r_t - \delta_k - g = \rho\left(1+g\right), \\
c_t^d &= \frac{1}{1+\alpha}\left[w_t h_t\left(1 - l_{Ht}\right) + k_t \rho\left(1+g\right)\right].
\end{aligned}$$

Solving for Human Capital Investment Time along BGP

- $$\begin{aligned} h_{t+1} &= h_t \left(1 + A_H l_{Ht} - \delta_H\right), \\ 1+g &= \frac{h_{t+1}}{h_t} = 1 + A_H l_{Ht} - \delta_H, \\ l_{Ht} &= \frac{g+\delta_H}{A_H}. \end{aligned}$$
- Substitute solution into consumption demand; normalize by h_t:

$$\begin{aligned} c_t^d &= \frac{1}{1+\alpha}\left[w_t h_t \left(1 - \frac{g+\delta_H}{A_H}\right) + k_t \rho \left(1+g\right)\right], \\ \frac{c_t^d}{h_t} &= \frac{1}{1+\alpha}\left[w_t \left(1 - \frac{g+\delta_H}{A_H}\right) + \frac{k_t}{h_t} \rho \left(1+g\right)\right]. \end{aligned}$$

- Consumption demand given as function of
  - parameters, stationary endogenous g, w_t, state variable $\frac{k_t}{h_t}$.
  - Given g and $\frac{k_t}{h_t}$, can graph as a function of $\frac{1}{w_t}$ as before.
  - Growth rate g determined by consumer choice of l_{Ht}.

Permanent Income and Wealth

$$c_t^d = \left(\frac{1}{1+\alpha}\right) y_{Pt};$$
$$y_{Pt} = w_t h_t \left(1 - \frac{g+\delta_H}{A_H}\right) + k_t \rho \left(1+g\right).$$

$$T_t = l_t^s + x_t = 1 - \frac{g+\delta_H}{A_H};$$
$$y_{Pt} = w_t h_t T_t + k_t \rho \left(1+g\right).$$

$$W_t = \frac{y_{Pt}}{R_t^W} = \frac{w_t h_t T_t + k_t \rho \left(1+g\right)}{R_t^W}, \; R_t^W \equiv \rho\left(1+g\right),$$
$$W_t = \frac{w_t h_t T_t}{\rho\left(1+g\right)} + k_t.$$

- Wealth equals discounted human capital earnings stream $\frac{w_t h_t T_t}{\rho(1+g)}$,
  - plus physical capital.

Aggregate Demand AD

$$\begin{aligned} y_t^d &= c_t^d + [k_{t+1} - k_t(1-\delta_k)] = c_t^d + k_t(g+\delta_k), \\ y_t^d &= \frac{1}{1+\alpha}\left[w_t h_t\left(1 - \frac{g+\delta_H}{A_H}\right) + k_t[\rho(1+g) + (g+\delta_k)(1+\alpha)]\right], \\ \frac{y_t^d}{h_t} &= \frac{1}{1+\alpha}\left[w_t\left(1 - \frac{g+\delta_H}{A_H}\right) + \frac{k_t}{h_t}[\rho(1+g) + (g+\delta_k)(1+\alpha)]\right]. \end{aligned}$$

$$\frac{1}{w_t} = \frac{1 - \frac{g+\delta_H}{A_H}}{\frac{y_t^d}{h_t}(1+\alpha) - \frac{k_t}{h_t}[\rho(1+g) + (g+\delta_k)(1+\alpha)]}$$

- Can normalize $h_t = 1$:

$$\frac{1}{w_t} = \frac{1 - \frac{g+\delta_H}{A_H}}{y_t^d(1+\alpha) - k_t[\rho(1+g) + (g+\delta_k)(1+\alpha)]}$$

Aggregate Supply AS

$$
\begin{aligned}
\underset{l_t^d, k_t}{Max}\ \Pi_t &= y_t^s - w_t l_t^d h_t - r_t k_t = A_G \left(l_t^d h_t\right)^{\gamma} (k_t)^{1-\gamma} - w_t l_t^d h_t - r_t k_t \\
w_t &= \gamma A_G \left(l_t^d h_t\right)^{\gamma-1} (k_t)^{1-\gamma}, \\
r_t &= (1-\gamma) A_G \left(l_t^d h_t\right)^{\gamma} (k_t)^{-\gamma}.
\end{aligned}
$$

$$
\begin{aligned}
l_t^d &= \left(\frac{\gamma A_G}{w_t}\right)^{\frac{1}{1-\gamma}} \frac{k_t}{h_t},\ y_t^s = A_G \left(l_t^d h_t\right)^{\gamma} (k_t)^{1-\gamma}, \\
y_t^s &= A_G \left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}} k_t,\ \frac{y_t^s}{h_t} = A_G \left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}} \frac{k_t}{h_t}.
\end{aligned}
$$

$$
\frac{1}{w_t} = \frac{1}{\gamma A_G} \left(\frac{y_t^s}{A_G k_t}\right)^{\frac{1-\gamma}{\gamma}}.
$$

- Same *AS* function as Chapter 8 exogenous growth.

Example 12.1 Baseline Calibration

- Target $g = 0.02$, as in exogenous growth Chapter 11.
- Appendix $A12$ gives methodology for solving g.
- $\gamma = \frac{1}{3}$, $\alpha = 1$, $A_h = 0.189$, $\delta_k = 0.05$, $\delta_h = 0.015$, $\beta = \frac{1}{1+\rho} = 0.95$, $(\rho = \frac{1}{0.95} - 1 = 0.052632)$, , $A_G = 0.28224$.
  - $\Longrightarrow g = 0.020$, $\frac{k_t}{h_t} = 1.00$. With $h_t = 1$, then also $k_t = 1$.

Example 12.1 Equilibrium AS-AD

- Other equilibrium variables: $l_H = 0.185\,26$, $l = 0.28405$, $1 - x = 0.185\,26 + 0.284\,05 = 0.469\,31$.
  - $x_t = 0.531$, close to Gomme and Rupert (2007).
  - $r = 0.1237$, or 12.37%, and $w = 0.21772$; $\frac{1}{w_t} = 4.59$.
- Return on human capital, Zhang (2006): 9.6% − 10.6%.
  - Here return is $A_H(1 - x_t) = r_t - \delta_k + \delta_h$
  - $= 0.1237 - 0.05 + 0.015 = 0.0887$, in Zhang range.
- $AD - AS$ calibrated functions:

$$\frac{1}{w_t} = \frac{\left(1 - \frac{0.02+0.015}{0.189}\right)}{y(1+1) - 1\left[(0.052632)(1+0.02) + (0.02+0.05)(1+1)\right]};$$

$$\frac{1}{w_t} = \frac{3}{0.282\,24}\left(\frac{1}{(0.282\,24)\,1}\right)^2 y_t^2.$$

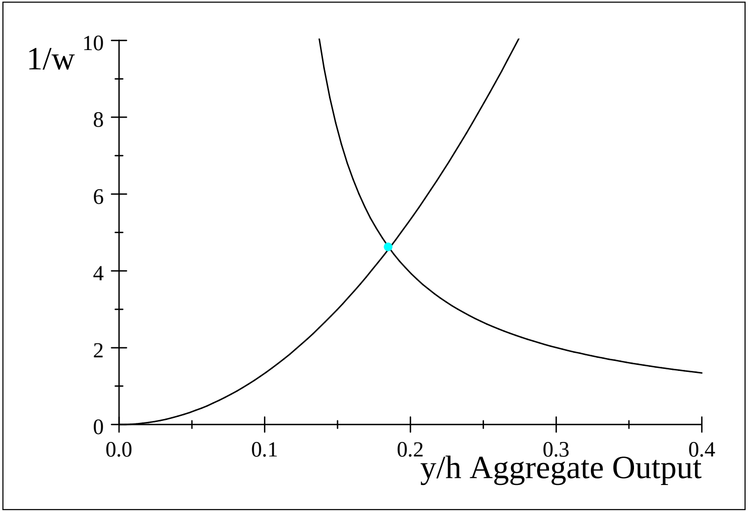


Figure 12.1. $AS - AD$ with Human Capital and Endogenous Growth in Example 12.1.

Excess Output Demand with Human Capital

$$
\begin{aligned}
0 &= Y(w_t) = y_t^d - y_t^s, \\
&= \frac{\left[w_t h_t \left(1 - \frac{g+\delta_H}{A_H}\right) + k_t \left[\rho(1+g) + (g+\delta_k)(1+\alpha)\right]\right]}{1+\alpha} \\
&\quad -A_G \left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}} k_t
\end{aligned}
$$

$$
\begin{aligned}
0 &= Y(w_t) = \\
&\frac{\left(w_t \left(1 - \frac{0.02+0.015}{0.189}\right) + 1\left(0.0526(1.02) + (0.02+0.05)(2)\right)\right)}{2} \\
&-0.28224 \left(\frac{0.28224}{3(w_t)}\right)^{0.5} (1).
\end{aligned}
$$

Excess Output Demand Y(w)

$w = 0.2177$

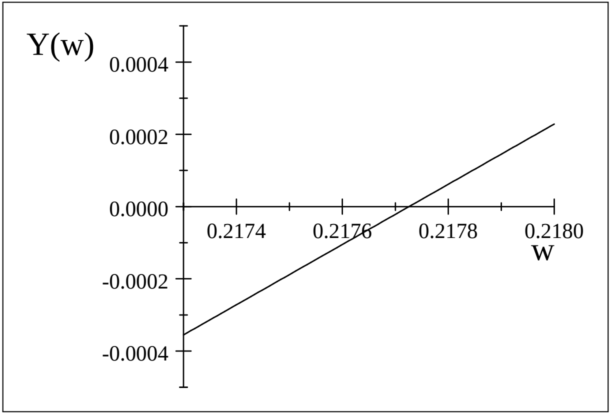


Figure 12.2. Excess Output Demand with Baseline Endogenous Growth in Example 12.1.

Consumption and Output

$$c_t^d = \frac{1}{1+\alpha}\left[w_t h_t\left(1-\frac{g+\delta_H}{A_H}\right)+k_t\rho\left(1+g\right)\right],$$

$$c_t^d = \frac{1}{1+1}\left(0.217\,72\left(1-\frac{0.020+0.015}{0.189}\right)+0.05\,26\left(1.020\right)\right)$$

$$= 0.115\,53.$$

$$y_t^d = \frac{\left[w_t h_t\left(1-\frac{g+\delta_H}{A_H}\right)+k_t\left[\rho\left(1+g\right)+\left(g+\delta_k\right)\left(1+\alpha\right)\right]\right]}{1+\alpha},$$

$$y_t^d = \frac{\left(0.217\,72\left(1-\frac{0.02+0.015}{0.189}\right)+1\left(0.0526\left(1.02\right)+\left(0.07\right)\left(2\right)\right)\right)}{2}$$

$$= 0.18553.$$

$$\frac{c_t^d}{y_t^d} = \frac{0.115\,53}{0.185\,53} = 0.6227;$$

Isocost, Isoquant, and Input Ratio with Human Capital

$$
\begin{aligned}
\textit{Isocost} \quad &: \quad y_t^s = w_t l_t^d h_t + r_t k_t, \\
\frac{0.185\,53}{h_t} &= (0.217\,7)\, l_t^d + (0.1237)\, \frac{k_t}{h_t}, \\
\frac{k_t}{h_t} &= \frac{0.185\,53}{0.1237 h_t} - \frac{(0.217\,7)\, l_t^d}{0.1237}.
\end{aligned}
$$

$$
\begin{aligned}
\textit{Isoquant} \quad &: \quad y_t^s = A_G \left(l_t^d h_t \right)^{\gamma} (k_t)^{1-\gamma}, \\
0.185\,53 &= (0.282\,24) \left(l_t^d h_t \right)^{\frac{1}{3}} (k_t)^{\frac{2}{3}}; \\
\frac{k_t}{h_t} &= \left(\frac{(0.185\,53)}{(0.282\,24)\, h_t \left(l_t^d \right)^{\frac{1}{3}}} \right)^{\frac{3}{2}} = \frac{\left(\frac{0.185\,53}{(0.282\,24) h_t} \right)^{\frac{3}{2}}}{\left(l_t^d \right)^{\frac{1}{2}}}.
\end{aligned}
$$

$$
\textit{Input Ratio} : \frac{k_t}{h_t} = \frac{1}{0.284\,05} l_t^d = (3.5205)\, l_t^d.
$$

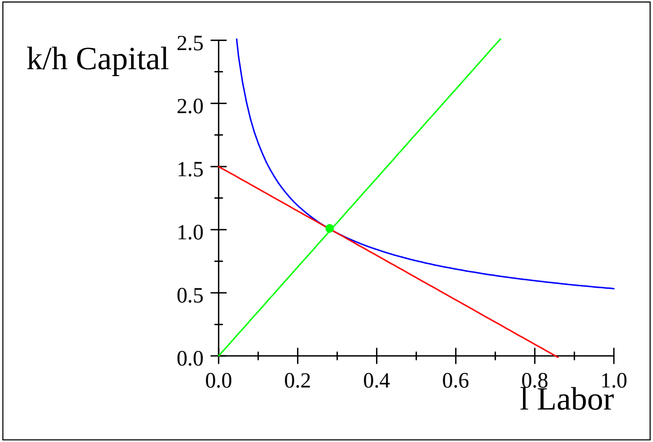


Figure 12.3. Factor Market Equilibrium with Baseline Endogenous Growth in Example 12.1.

Production, Utility Level, Budget Line with Human Capital

Production

$$\begin{aligned} c_t^d &= y_t^s - i_t = A_G \left(l_t^d h_t \right)^{\gamma} (k_t)^{1-\gamma} - (g + \delta_k) k_t, \\ \frac{c_t^d}{h_t} &= (0.282\,24) \left(l_t^d \right)^{\frac{1}{3}} (1)^{\frac{2}{3}} - (0.02 + 0.05)(1). \end{aligned}$$

Utility

$$\begin{aligned} u &= \ln c_t^d + \alpha \ln x_t = \ln c_t^d + \alpha \ln \left(1 - l_{Ht} - l_t^s \right), \\ -2.791\,2 &= \ln 0.115\,53 + 1 \ln (0.531), \\ -2.791\,2 &= \ln c_t^d + \ln \left(T_t - l_t^s \right), \\ c_t^d &= \frac{e^{-2.791\,2}}{\left(1 - 0.185\,26 - l_t^s\right)}. \end{aligned}$$

Budget line

$$c_t^d = w_t l_t^s h_t + \rho (1 + g) k_t^s = (0.217\,7) l_t^s + 0.0526 (1.02).$$

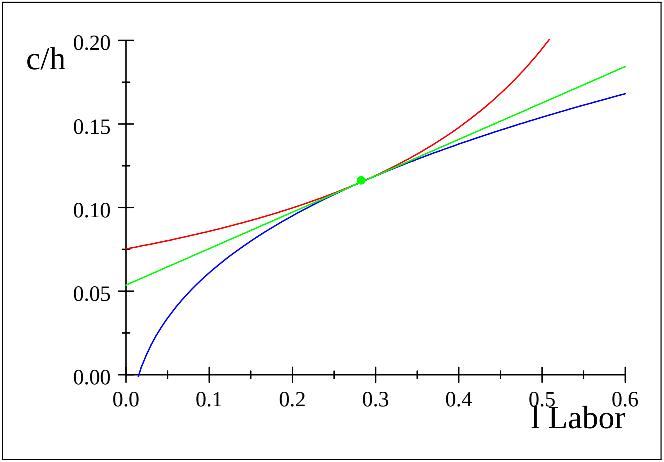


Figure 12.4. General Equilibrium Goods and Labor with Baseline Endogenous Growth in Example 12.1.

Labor Supply, Demand with Human Capital

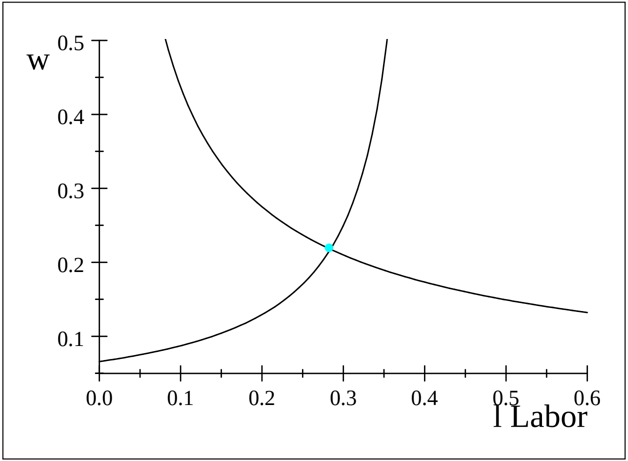
$$
\begin{aligned}
\frac{c_t^d \alpha}{w_t h_t} &= x_t = 1 - l_t^s - l_{Ht}, \\
\frac{c_t^d}{h_t} &= \frac{1}{1+\alpha}\left[w_t\left(1 - l_{Ht}\right) + \frac{k_t}{h_t}\rho\left(1+g\right)\right], \\
l_t^s &= 1 - \frac{c_t^d \alpha}{w_t h_t} - l_{Ht};\ l_{Ht} = \frac{g+\delta_H}{A_H}, \\
&\Rightarrow\ l_t^s = 1 - \frac{\alpha}{1+\alpha}\left[1 + \frac{k_t}{w_t h_t}\rho\left(1+g\right)\right] - \frac{g+\delta_H}{A_H\left(1+\alpha\right)}.
\end{aligned}
$$

$$
\begin{aligned}
w_t &= \gamma A_G \left(l_t^d h_t\right)^{\gamma-1}\left(k_t\right)^{1-\gamma}, \\
l_t^d &= \left(\frac{\gamma A_G}{w_t}\right)^{\frac{1}{1-\gamma}}\frac{k_t}{h_t};\ h_t = 1,\ \Rightarrow l_t^d = \left(\frac{\gamma A_G}{w_t}\right)^{\frac{1}{1-\gamma}} k_t,
\end{aligned}
$$

Labor Supply and Demand in Example 12.1

$$
\begin{aligned}
w_t &= \frac{\alpha\rho\left(1+g\right)\left(\frac{k_t}{h_t}\right)}{1-\left(1+\alpha\right)l_t^s-\frac{\left(g+\delta_H\right)}{A_H}}; \\
w_t &= \gamma A_G\left(\frac{k_t}{h_t l_t^d}\right)^{1-\gamma}.
\end{aligned}
$$

$$
\begin{aligned}
w_t &= \frac{1\left(0.0526\right)\left(1+0.02\right)}{1-\left(1+1\right)l_t^s-\frac{\left(0.02+0.015\right)}{0.189}}; \\
w_t &= \frac{\left(0.282\,24\right)}{3}\frac{\left(1\right)^{\frac{2}{3}}}{\left(l_t^d\right)^{\frac{2}{3}}}.
\end{aligned}
$$



Fitgure 12.5. Labor Market with Endogenous Growth Baseline Model in Example 12.1.

Excess Labor Demand L(w) with Human Capital

$$0 = L(w_t) = l_t^d - l_t^s = \left(\frac{\gamma A_G}{w_t}\right)^{\frac{1}{1-\gamma}} \frac{k_t}{h_t} - \left(1 - \frac{\alpha}{1+\alpha}\left[1 + \frac{k_t}{w_t h_t}\rho(1+g)\right] - \frac{g+\delta_H}{A_H(1+\alpha)}\right).$$

$$0 = L(w) = \left(\frac{0.282\,24}{3w_t}\right)^{1.5} 1 - \left(1 - \frac{1}{1+1}\left(1 + \frac{0.0526(1+0.02)}{w_t}\right) - \frac{0.02+0.015}{0.189(1+1)}\right).$$

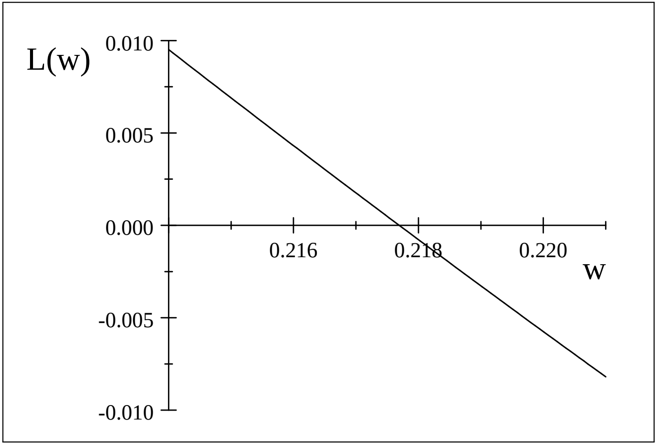


Figure 12.6. Excess Labor Demand in Baseline Endogenous Growth Model

Endogenous Growth Process

- c_t, y_t, k_t, h_t, i_t, grow at rate g; parameters stationary.
- Inversely AD in $\left(\frac{1}{w_t} : y\right)$ space is stationary:

$$\frac{1}{w_t} = \frac{h_t\left(1 - \frac{g+\delta_H}{A_H}\right)}{y_t^d\left(1+\alpha\right) - k_t\left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left(1+\alpha\right)\right]}.$$

- However $\frac{1}{w_t h_t}$ is nonstationary:

$$\frac{1}{w_t h_t} = \frac{1 - \frac{g+\delta_H}{A_H}}{y_t^d\left(1+\alpha\right) - k_t\left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left(1+\alpha\right)\right]}.$$

  - Shift out over time, as effective wage rises.
  - Supply in effective wage also nonstationary:

$$\frac{1}{w_t h_t} = \left(\frac{y_t^s}{k_t A_G}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\gamma A_G h_t}\frac{k_t}{k_t}$$

$$\frac{1}{w_t h_t} = \left(\frac{y_t^s}{k_t A_G}\right)^{\frac{1-\gamma}{\gamma}} \frac{k_t}{h_t}\frac{1}{\gamma A_G k_t}$$

- $\alpha = 1$, $A_h = 0.189$, $\delta_k = 0.05$, $\delta_h = 0.015$, $\beta = \frac{1}{1+\rho} = 0.95$, $\rho = \frac{1}{0.95} - 1 = 0.0526$, $A_G = 0.282\,24$; $g = 0.020$, $\frac{k_t}{h_t} = 1.00$.
  - $k_{t+1} = k_t\left(1+g\right)$; $h_{t+1} = h_t\left(1+g\right)$
  - $k_t = 1$, $k_{t+1} = 1.02$; $h_t = 1$, $h_{t+1} = 1.02$.
- Consider $t+1 : AD, AS$

$$\frac{1}{w_{t+1}h_{t+1}} = \frac{1 - \frac{g+\delta_H}{A_H}}{y^d_{t+1}\left(1+\alpha\right) - k_{t+1}\left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left(1+\alpha\right)\right]};$$

$$\frac{1}{w_{t+1}h_{t+1}} = \left(\frac{y^s_{t+1}}{k_{t+1}A_G}\right)^{\frac{1-\gamma}{\gamma}} \frac{k_{t+1}}{h_{t+1}} \frac{1}{\gamma A_G k_{t+1}};$$

Writing k_{t+1} and h_{t+1} in terms of time t, using the 2% stationary growth whereby $k_{t+1} = k_t\left(1+g\right)$ and $h_{t+1} = h_t\left(1+g\right)$, these functions can be written as

Growth in Effective Wage and Output

$$\frac{1}{w_{t+1}h_{t+1}} = \frac{1-\frac{g+\delta_H}{A_H}}{y^d_{t+1}(1+\alpha)-k_t(1+g)\left[\rho(1+g)+(g+\delta_k)(1+\alpha)\right]},$$

$$\frac{1}{w_{t+1}h_{t+1}} = \left(\frac{y^s_{t+1}}{k_t(1+g)A_G}\right)^{\frac{1-\gamma}{\gamma}}\frac{k_t(1+g)}{h_t(1+g)}\frac{1}{\gamma A_G k_t(1+g)};$$

$$\frac{1}{w_{t+2}h_{t+2}} = \frac{1-\frac{g+\delta_H}{A_H}}{y^d_{t+2}(1+\alpha)-k_t(1+g)^2\left[\rho(1+g)+(g+\delta_k)(1+\alpha)\right]},$$

$$\frac{1}{w_{t+2}h_{t+2}} = \left(\frac{y^s_{t+2}}{k_t(1+g)^2A_G}\right)^{\frac{1-\gamma}{\gamma}}\frac{k_t(1+g)^2}{h_t(1+g)^2}\frac{1}{\gamma A_G k_t(1+g)^2}.$$

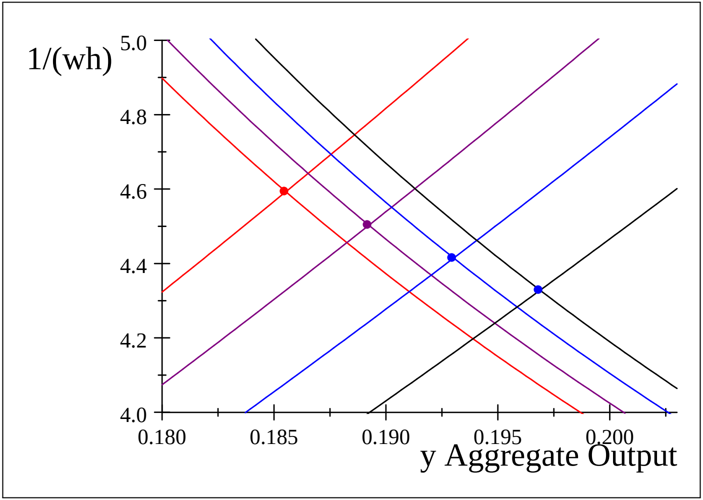
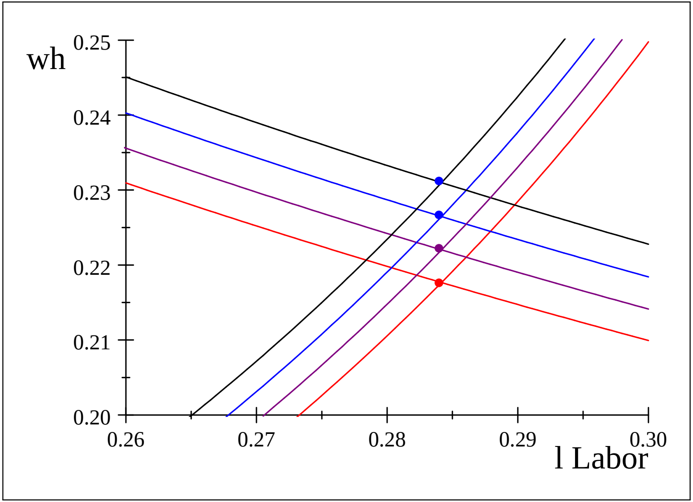


Figure 12.7. Endogenous Growth shifts in $AS - AD$ from time t to $t + 3$ in Example 12.2.

Labor Market Endogenous Growth Process

$$
\begin{aligned}
w_t h_t &= \frac{\alpha\rho\,(1.02)\,(k_t)}{1-(1+\alpha)\,l_t^s - \frac{(g+\delta_H)}{A_H}}, \quad w_t h_t = \frac{\left(\frac{h_t}{k_t}\right)^{\gamma}}{\frac{1}{\gamma A_G}\left(l_t^d\right)^{1-\gamma}} k_t; \\
w_{t+1} h_{t+1} &= \frac{\alpha\rho\,(1.02)^2\,(k_t)}{1-(1+\alpha)\,l_{t+1}^s - \frac{(g+\delta_H)}{A_H}}; \quad w_{t+1} h_{t+1} = \frac{\left(\frac{h_t}{k_t}\right)^{\gamma} k_t\,(1.02)}{\frac{1}{\gamma A_G}\left(l_{t+1}^d\right)^{1-\gamma}} \\
w_{t+2} h_{t+2} &= \frac{\alpha\rho\,(1.02)^3\,(k_t)}{1-(1+\alpha)\,l_{t+2}^s - \frac{(g+\delta_H)}{A_H}}; \quad w_{t+2} h_{t+2} = \frac{\left(\frac{h_t}{k_t}\right)^{\gamma} k_t\,(1.02)^2}{\frac{1}{\gamma A_G}\left(l_{t+2}^d\right)^{1-\gamma}} \\
w_{t+3} h_{t+3} &= \frac{\alpha\rho\,(1.02)^4\,(k_t)}{1-(1+\alpha)\,l_{t+3}^s - \frac{(g+\delta_H)}{A_H}}; \quad w_{t+3} h_{t+3} = \frac{\left(\frac{h_t}{k_t}\right)^{\gamma} k_t\,(1.02)^3}{\frac{1}{\gamma A_G}\left(l_{t+3}^d\right)^{1-\gamma}}
\end{aligned}
$$



Fitgure 12.8. Labor Market with Endogenous Growth Baseline Model in Example 12.2.

Example 12.3 Labor Intensity Increase

- $\gamma = \frac{2}{3}$; $A_h = 0.154$, $A_G = 0.749\,55$, $\alpha = 1$, $\delta_k = 0.05$, $\delta_h = 0.015$, $\beta = 0.95$.
- $\frac{k}{h} = 1$, $g = 0.0200$, $l = 0.3486$; $l_H = 0.2274$, $1 - x = 0.2274 + 0.3486 = 0.576$, $w = 0.710\,12$.

$$\frac{1}{w_t} = \frac{\left(1 - \frac{0.02+0.015}{0.154}\right)}{y^d\left(1+1\right) - 1\left(\left(0.0526\right)\left(1+0.02\right) + \left(0.07\right)\left(1+1\right)\right)};$$

$$\frac{1}{w_t} = \frac{1}{\left(0.667\right)0.749\,55}\left(\frac{1}{\left(0.749\,55\right)1}\right)^{\frac{1-(0.667)}{(0.667)}}\left(y_t^s\right)^{\frac{1-(0.667)}{(0.667)}}.$$

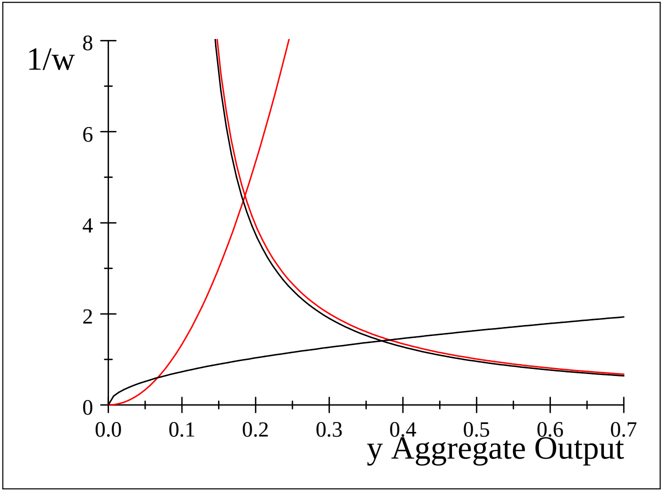


Figure 12.9. $AS - AD$ with Human Capital and High Labor Intensity in Example 12.3.

Special Cases of Endogenous Growth

Example 12.4: Human Capital Only, Linear Production

$$
\begin{aligned}
c_t &= y_t = A_G l_t h_t;\ i_{ht} = A_H l_{Ht} h_t,\ h_{t+1} = h_t\left(1-\delta_h\right) + i_t; \\
V\left(h_t\right) &= \\
\underset{l_t, l_{Ht}}{Max} &: \ \ln\left(w_t l_t h_t\right) + \alpha \ln\left(1 - l_t - l_{Ht}\right) + \beta V\left[h_t\left(1-\delta_h\right) + A_H l_{Ht} h_t\right]. \\
\frac{\alpha c_t}{x_t} &= w_t h_t,\ 1+g = \frac{1 + A_H\left(1-x\right) - \delta_h}{1+\rho},\ x = \frac{\alpha\rho}{A_H}\left[\frac{1 + A_H - \delta_h}{1+\rho\left(1+\alpha\right)}\right]. \\
g &= \frac{A_H - \delta_h - \alpha\rho\left[\frac{1+A_H-\delta_h}{1+\rho(1+\alpha)}\right] - \rho}{1+\rho},\ l_H = \frac{g+\delta_H}{A_H}; \\
l_H &= \frac{1 + \frac{\alpha\rho}{1+\rho(1+\alpha)}}{1+\rho} - \frac{\rho\left(1-\delta_H\right) - \frac{\alpha\rho(1-\delta_h)}{1+\rho(1+\alpha)}}{A_H\left(1+\rho\right)};\ \frac{\partial l_H}{\partial A_H} > 0.
\end{aligned}
$$

Example 12.5: Physical Capital Only

$$
\begin{aligned}
y_t &= A_G k_t,\ c_t = r_t k_t - k_{t+1} + k_t\left(1-\delta_k\right),\ u_t = \ln c_t, \\
V\left(k_t\right) &= \underset{k_{t+1}}{Max} : \ln\left(r_t k_t - k_{t+1} + k_t\left(1-\delta_k\right)\right) + \beta V\left(k_{t+1}\right); \\
1+g &= \frac{c_t}{c_{t-1}} = \beta\left(1 + r_t - \delta_k\right),\ r_t = A_G; \\
g &= \frac{A_G - \delta_k - \rho}{1+\rho}.
\end{aligned}
$$

- "Ak" economy called endogenous growth
  - but trivially determined by parameter assumptions
  - A_G, δ_k, ρ.
- Equivalent to human only economy
  - when leisure preference α is zero,
  - and $A_H = A_G$ and $\delta_h = \delta_k$.

Applications: Income Distribution, Marx, and Human Capital

- Thorstein Veblen 1899 Theory of the Leisure Class
  - Low income classes emulate higher income classes
  - through "conspicuous consumption".
- Marx's 1867 application of classical labor theory
  - rigidly divide classes into worker, capitalist,
  - forcibly give capitalist income to workers;
  - a theory built on maintaining inter-class tensions.
  - Requires centralized government to redistribute.
- Alternative theory of marginal productivity:
  - attaining education gives theory of social stability,
  - with upward class movement, decentralized society.
  - Allows anyone to own small shares of capital;
  - merged capital/labor class, depending on shares.
- After Veblen, human capital theory of TW Schultz.
  - emulation occurs as low income get skills/education.

Appendix A12 Solution Methodology

- Excess output demand $Y(w_t, h_t, k_t, g)$ equal to zero.
- Reformulate as function of only growth rate g :
  - Target $g = 0.02$ as solution to this equation.
  - Solve $Y(g)$ by solving $\frac{k_t}{h_t}(g)$ and $w_t(g)$.
- Gives a quadratic equation in g.
  - Can solve graphically,
  - analytically using quadratic solution.
- Solution methodology uses 3 intertemporal margins:
  - from physical capital return,
  - human capital return,
  - human capital investment function along BGP.

Solving for Growth Rate with Excess Goods Demand

$$
\begin{aligned}
Y\left(w_t, h_t, k_t, g\right) &= \frac{w_t h_t}{1+\alpha}\left(1-\frac{g+\delta_H}{A_H}\right)+k_t\left(\frac{\rho\left(1+g\right)}{1+\alpha}+\left(g+\delta_k\right)\right) \\
&\quad -A_G\left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}} k_t = 0.
\end{aligned}
$$

$$
\begin{aligned}
1+g &= \beta\left(1+r_t-\delta_k\right)=\beta\left[1+A_H\left(1-x_t\right)-\delta_H\right]=1+A_H l_{Ht}-\delta_H; \\
&\Longrightarrow\; 1-x_t=\frac{\left(1-\delta_H\right)\left(1-\beta\right)+g+\delta_H}{A_H\beta},\; l_t=\frac{\left(1+g\right)\left(1-\beta\right)}{A_H\beta}; \\
r_t &= \left(1-\gamma\right)A_G\left(\frac{k_t}{h_t}\right)^{-\gamma}\left(l_t\right)^{\gamma},\; \frac{k_t}{h_t}=\left(\frac{\left(1-\gamma\right)A_G}{r_t}\right)^{\frac{1}{\gamma}} l_t, \\
\frac{k_t}{h_t} &= \left(\frac{\beta\left(1-\gamma\right)A_G}{1+g+\beta\left(\delta_k-1\right)}\right)^{\frac{1}{\gamma}}\frac{\left(1+g\right)\left(1-\beta\right)}{A_H\beta};\; w\left(\frac{k_t}{h_t}, l_t\right).
\end{aligned}
$$

Solving for the Growth Rate from AS-AD

$$0 = \frac{1}{1+\alpha}\left[\left(1-\frac{g+\delta_H}{A_H}\right)+\frac{k_t}{w_t h_t}\left[\rho\left(1+g\right)+\left(g+\delta_k\right)\left(1+\alpha\right)\right]\right]$$
$$-A_G\left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}}\frac{k_t}{w_t h_t};$$

$$0 = \frac{\left(1-\frac{g+\delta_H}{A_H}\right)+\frac{(1+g)(1-\beta)(1-\gamma)}{(1+g+\beta(\delta_k-1))A_H\gamma}\left[\rho\left(1+g\right)+\left(g+\delta_k\right)\left(1+\alpha\right)\right]}{1+\alpha}$$
$$-\frac{\left[1+g+\beta\left(\delta_k-1\right)\right]}{\beta\left(1-\gamma\right)}\frac{\left(1+g\right)\left(1-\beta\right)\left(1-\gamma\right)}{\left(1+g+\beta\left(\delta_k-1\right)\right)A_H\gamma};$$

$$0 = \beta\left(A_H-g-\delta_H\right)\left(1+g+\beta\left(\delta_k-1\right)\right)\gamma$$
$$+\beta\left(1+g\right)\left(1-\beta\right)\left(1-\gamma\right)\left[\rho\left(1+g\right)+\left(g+\delta_k\right)\left(1+\alpha\right)\right]$$
$$-\left(1+\alpha\right)\left(1+g\right)\left(1-\beta\right)\left[1+g+\beta\left(\delta_k-1\right)\right].$$

Example 12.1

$\alpha = 1$, $A_h = 0.189$, $\delta_k = 0.05$, $\delta_h = 0.015$, $\beta = \frac{1}{1+\rho} = 0.95$,
$\rho = \frac{1}{0.95} - 1 = 0.0526$, $A_G = (0.28224)$, $\gamma = \frac{1}{3} \Longrightarrow \frac{k_t}{h_t} = 1$; $g = 0.02$.

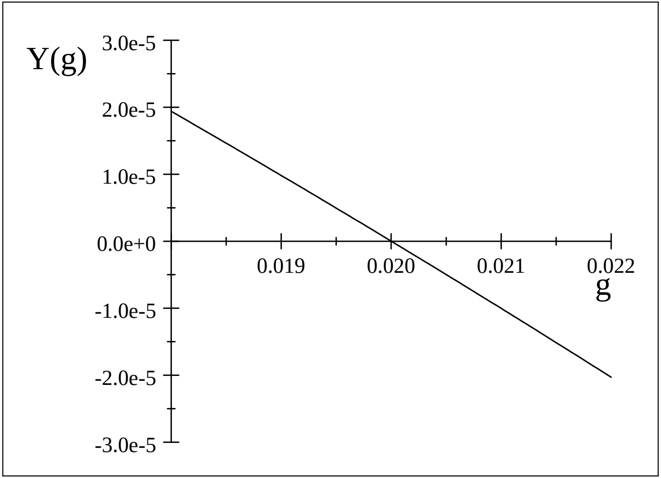


Figure 12.10. Normalized Baseline Excess Demand $Y(g)$.

Example 12.3

$\gamma = \frac{2}{3}$; $\alpha = 1$, $A_h = 0.154$, $\delta_h = 0.015$, $\beta = 0.95$,
$A_G = 0.749\,55 \Longrightarrow \frac{k}{h} = 1.00$; $g = 0.020$.

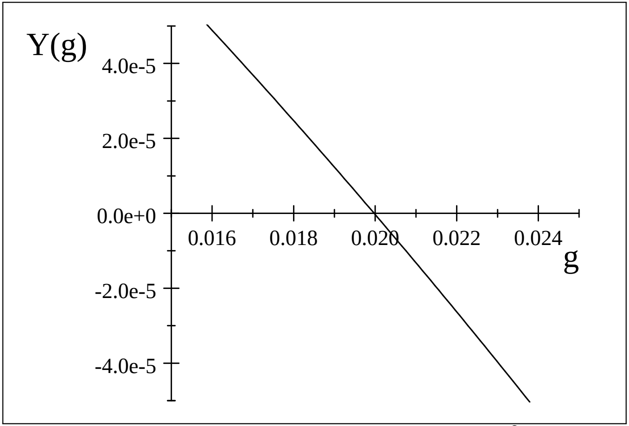


Figure 12.11. Normalized Excess Demand, with $\gamma = \frac{2}{3}$, as a Function of g in Example 12.3.

Quadratic Equation Solution of Growth Rate

$$
\begin{aligned}
0 &= \beta\gamma\left(A_h - \delta_h - g\right)\left[1 + \beta\left(\delta_k - 1\right) + g\right] \\
&\quad + \beta\left(1-\beta\right)\left(1-\gamma\right)\left[\rho + \delta_k\left(1+\alpha\right) + \left(1+\alpha+\rho\right)g\right]\left(1+g\right) \\
&\quad - \left(1+\alpha\right)\left(1-\beta\right)\left[1 + \beta\left(\delta_k - 1\right) + g\right]\left(1+g\right); \\
0 &= Ag^2 + Bg + C,\ g = \frac{-B - \sqrt{B^2 - 4AC}}{2A}
\end{aligned}
$$

$$
\begin{aligned}
A &\equiv -\beta\gamma + \beta\left(1-\beta\right)\left(1-\gamma\right)\left(1+\alpha+\rho\right) - \left(1+\alpha\right)\left(1-\beta\right), \\
B &\equiv -\beta\gamma\left[1 + \beta\left(\delta_k - 1\right) - A_h + \delta_h\right] - \left(1+\alpha\right)\left(1-\beta\right)\left[2 + \beta\left(\delta_k - 1\right)\right] \\
&\quad + \beta\left(1-\beta\right)\left(1-\gamma\right)\left[\rho + \delta_k\left(1+\alpha\right) + 1 + \alpha + \rho\right], \\
C &\equiv \beta\gamma\left(A_h - \delta_h\right)\left[1 + \beta\left(\delta_k - 1\right)\right] + \beta\left(1-\beta\right)\left(1-\gamma\right)\left[\rho + \delta_k\left(1+\alpha\right)\right] \\
&\quad - \left(1+\alpha\right)\left(1-\beta\right)\left[1 + \beta\left(\delta_k - 1\right)\right].
\end{aligned}
$$

- Example 12.1 : $\alpha = 1$, $A_h = 0.189$, $\delta_k = 0.05$, $\delta_h = 0.015$, $\beta = \frac{1}{1+\rho} = 0.95$, $\rho = \frac{1}{0.95} - 1 = 0.0526$, $A_G = (0.282\,24)$, $\gamma = \frac{1}{3}$:

$$\begin{aligned} \text{Quadratic term:} \quad & A = -0.3517, \\ \text{Linear term:} \quad & B = -0.0157, \\ \text{Constant term:} \quad & C = 0.00045558 \end{aligned}$$

  - $g = 0.020$ for positive root;
  - negative root -0.0893 not used; violates $l_{Ht} \geq 0$.

- Example 12.3 : $\gamma = \frac{2}{3}$; and $\alpha = 1$, $A_h = 0.154$, $\delta_h = 0.015$, $\beta = 0.95$, $A_G = 0.74955$.

$$\begin{aligned} \text{Quadratic term:} \quad & A = -0.7008, \\ \text{Linear term:} \quad & B = -0.0486, \\ \text{Constant term:} \quad & C = 0.0012. \end{aligned}$$

  - Growth rate is same: $g = 0.020$.