Advanced Modern Macroeconomics

Explaining Cycles and Trends

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- Two main comparative static analyses:
  - goods sector, human capital productivity changes,
  - withing endogenous growth baseline economy,
- Describes business cycles, long term trends.
  - combines 2 comparative statics for business cycle
  - combines trends in same 2 parameters for trends.
- Increased goods productivity does not affect growth,
  - or employment, but increases wage rate, capital ratio.
- Increased human capital productivity increases growth,
  - decreases time left for work and leisure,
  - increases time in human capital investment.
  - Makes endogenous the change in time T.

Applications

- A look at shocks versus comparative statics.
- Growth accounting:
  - with endogenous versus exogenous growth;
  - equivalence on *BGP*.
- Trends explained in
  - historical growth rate, education time, working hours.
  - by slightly rising trend in human capital productivity.
- Symmetric growth models
  - with capacity utilization to compare to leisure;
  - plus Leontieff paradox,
  - and "structural transformation".

Building on the Last Chapters

- Example 12.2 used for comparative static exercises.
- Same solution methodology as in Chapter 12.
- Business cycle through comparative statics
  - similar to static Chapters 3 and 6, dynamic Chapter 9
  - but now most advanced, extended, dynamic model.
- Changing only human capital investment productivity
  - can explain the business cycle, unlike Chapters 3, 9.
  - Still can be combined with goods productivity change.
- Time endowment changed endogenously here
- versus exogenously in Chapters 3 and 9.
- Include explanation of trends, puzzles,
  - that cannot do with exogenous growth.

Learning Objective

- Seeing comparative statics with endogenous growth,
  - improves upon basic business cycles explanation,
  - with endogenous change in time for labor and leisure.
- How change in human capital productivity is plausible.
  - Shift between sectors during contraction,expansion.
  - From exogenous change in sector productivities.
  - Endogenous sector productivity change: beyond *RBC*.
- Endogenous growth for long term trends,
  - which otherwise elude explanation.
  - Simultaneous business cycle, long term trend power,
  - gives credence to endogenous growth approach.

Who Made It Happen

- Lucas 1977 set agenda for business cycles
  - stressing how variables move together during cycles.
  - Business cycles stylized facts, like Solow growth facts.
  - Set standard: 1975 "An Equilibrium Model of the Business Cycle,"
  - specified money supply as key for explaining cycles
  - following monetary approach of Friedman 1971.
- Kydland, Prescott 1982, Long, Plosser 1983
  - changes in goods productivity factor main effect.
  - Exploded open research in business cycles.
- *RBC* also led to new Keynesian approach
  - modified with price rigidities, monopoly power
  - Calvo 1983, Dixit, Stiglitz 1977
  - Lucas 2005: wage rigidity in first Kydland, Prescott 1987
  - paper but later dropped from published 1982 version.
- Lucas 1988 endogenous growth model
  - extension of real business cycle approach.

Lucas 2005 on Real Business Cycles:

"I call the conclusion that postwar US business cycles were mainly real in origin a discovery, but not because everyone working on business cycles at the time agreed with it. In fact, we all thought that there must be some other way to interpret Ed and Finn's results... But over the two decades since it appeared, every aspect of this paper has been gone over in detail, dozens of variations in the model have been explored indeed, many improvements have been found—and this central substantive finding has stood up."

Business Cycles with Endogenous Growth

- Explain business cycles within endogenous growth
  - also explain long term economic trends;
  - makes endogenous growth attractive.
- A_H change moves time l_{Ht} in same direction.
  - gives change in employment in opposite direction,
  - as is crucial to explain business cycle.
- A_H change also changes consumption to output ratio
  - in direction consistent with evidence.
  - Helps resolve a central "consumption-output" puzzle.
- A_G change: no effect on g, employment,
  - or consumption to output ratio.
  - But changes w, k/h, y, as in business cycle.

Example 13.1. Goods Sector Productivity Increase

- $\alpha = 1$, $A_H = 0.189$, $\delta_k = 0.05$, $\delta_h = 0.015$, $\beta = 0.95$.
- $A_G = 0.282\,24 \Longrightarrow A_G = 0.282\,24\,(1.05) = 0.29635$.
    - $\Longrightarrow g = 0.020$, unchanged; r unchanged;
    - both w, and $\frac{k}{h} = 1.1578$, are 16.78% higher.

$$\frac{1}{w_t} = \frac{1 - \frac{g+\delta_H}{A_H}}{\frac{y_t^d}{h_t}(1+\alpha) - \frac{k_t}{h_t}\left[\rho(1+g) + (g+\delta_k)(1+\alpha)\right]}$$

$$\frac{1}{w_t} = \frac{1}{\gamma A_G}\left(\frac{h_t}{A_G k_t}\right)^{\frac{1-\gamma}{\gamma}}\left(\frac{y_t}{h_t}\right)^{\frac{1-\gamma}{\gamma}};$$

$$\frac{1}{w_t} = \frac{\left(1 - \frac{0.02+0.015}{0.189}\right)}{y_t^d(1+1) - 1.1578\left((0.0526)(1+0.02) + (0.07)(2)\right)},$$

$$\frac{1}{w_t} = \frac{3}{0.29635}\left(\frac{1}{(0.29635)\,1.1578}\right)^2 (y_t^s)^2.$$

AS-AD Shift Out, Wage Rises

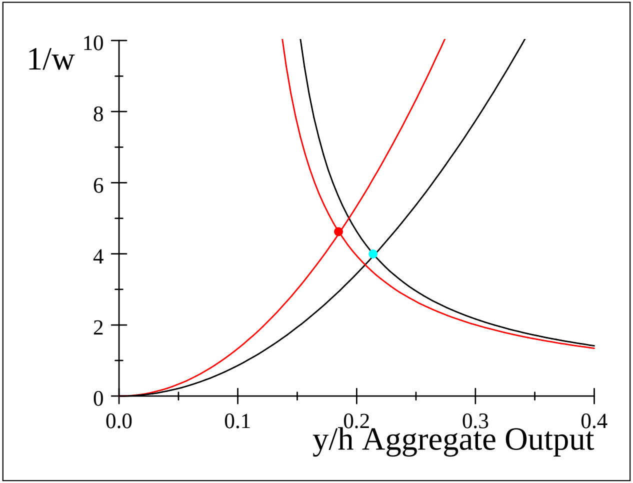


Figure 13.1. $AS - AD$ with Human Capital and Endogenous Growth in Example 13.1.

Finding Equilibrium Wage from Excess Demand

- Given that $k = 1.578$, can derive excess demand; graph.
- Where zero excess output demand, at equilibrium wage.

$$
\begin{aligned}
0 &= Y\left(w_t\right) = y_t^d - y_t^s, \\
&\frac{w_t\left(1 - \frac{0.02+0.015}{0.189}\right) + 1.1578\left[\left(\frac{1}{0.95} - 1\right)(1.02) + (0.07)(2)\right]}{2} \\
&-0.29635\left(\frac{0.29635}{3\left(w_t\right)}\right)^{0.5}(1.1578).
\end{aligned}
$$

Excess Demand with Goods Productivity Increase

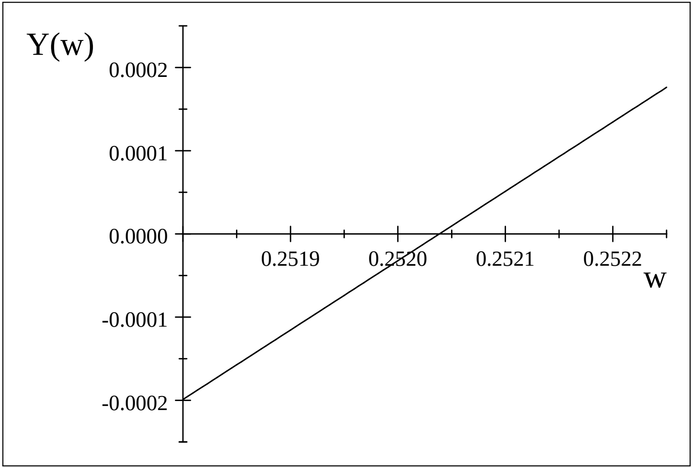


Figure 13.2. Excess Output Demand in Endogenous Growth with Goods Productivity Increase in Example 13.1.

$$
\begin{aligned}
c_t^d &= \\
&\frac{1}{1+1}\left(0.25206\left(1-\frac{0.020+0.015}{0.189}\right)+(1.1578)(0.0526)(1.020)\right) \\
&= 0.13375,
\end{aligned}
$$

$$
\begin{aligned}
y_t^d &= \\
&\frac{0.25206\left(1-\frac{0.02+0.015}{0.189}\right)+(1.1578)\left[\left(0.0526\left(1.02\right)+\left(0.07\right)\left(2\right)\right)\right]}{1+1} \\
&= 0.21480.
\end{aligned}
$$

$$
\frac{c_t^d}{y_t^d} = \frac{0.13375}{0.21480} = 0.622\,67.
$$

Labor Supply and Demand

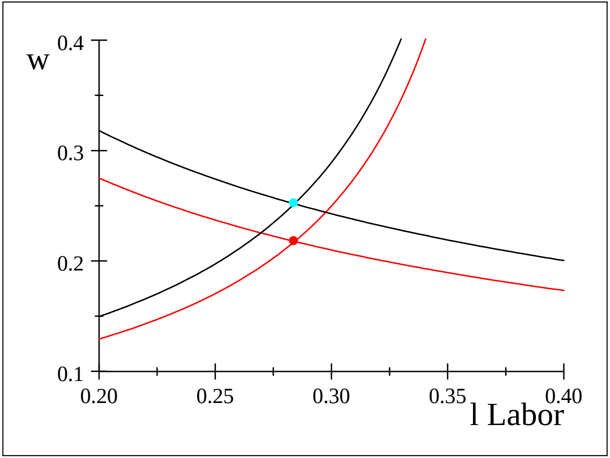
$$w_t h_t = \frac{\alpha\rho\left(1+g\right)\left(k_t\right)}{1-\left(1+\alpha\right)l_t^s - \frac{\left(g+\delta_H\right)}{A_H}};$$

$$w_t h_t = \gamma A_G \frac{\left(\frac{h_t}{k_t}\right)^{\gamma}}{\left(l_t^d\right)^{1-\gamma}} k_t.$$

$$w_t = \frac{\left(1.1578\right)\left(0.0526\right)\left(1+0.02\right)}{1-\left(1+1\right)l_t^s - \frac{(1+1)(0.02+0.015)}{0.189(1+1)}};$$

$$w_t = \frac{0.296}{3}\frac{\left(1.1578\right)^{\frac{2}{3}}}{\left(l_t^d\right)^{\frac{2}{3}}}.$$

Labor Market: Employment Unchanged



Fitgure 13.3. Labor Market with Endogenous Growth and a 5% Increase in A_G in Example 13.1.

Excess Labor Demand Function

$$
\begin{aligned}
0 &= L(w) = \left(\frac{0.29635}{3(0.25206)}\right)^{1.5}(1.1578) - \\
&\left[1 - \frac{1}{1+1}\left(1 + (1.1578)\frac{\left(\frac{1}{0.95} - 1\right)(1+0.02)}{0.25206}\right) - \frac{0.02+0.015}{0.189(1+1)}\right].
\end{aligned}
$$

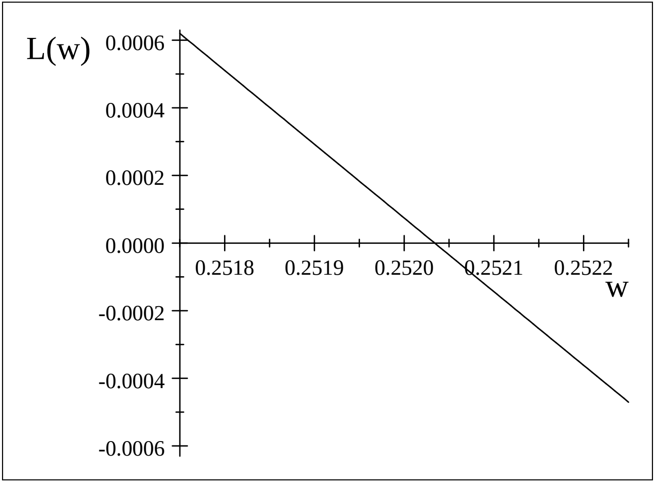


Figure 13.4. Excess Demand and the Real Wage when A_G increases 5% in Example 13.1.

Isocost, Isoquant, Input Ratio

Isocost

$$\begin{aligned} y_t &= w_t l_t h_t + r_t k_t = 0.21480, \\ \frac{k_t}{h_t} &= \frac{0.21480}{0.1237\,(1)} - \frac{(0.25206)\, l_t}{0.1237}. \end{aligned}$$

Isoquant

$$\begin{aligned} y_t^s &= A_G \left(l_t^d h_t \right)^{\gamma} (k_t)^{1-\gamma} = 0.21480, \\ \frac{k_t}{h_t} &= \left(\frac{0.21480}{(0.296\,35) \left(l_t^d \right)^{\frac{1}{3}}} \right)^{\frac{3}{2}} = \frac{\left(\frac{0.21480}{0.296\,35} \right)^{\frac{3}{2}}}{\left(l_t^d \right)^{\frac{1}{2}}}. \end{aligned}$$

Input Ratio

$$\frac{k_t}{h_t l_t} = \frac{1.1578}{0.284\,05} = 4.076.$$

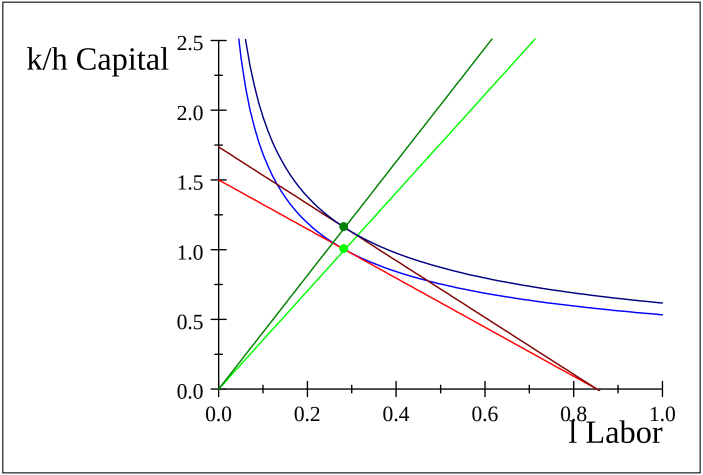


Figure 13.5. Factor Market Equilibrium with Baseline Endogenous Growth in Example 13.1.

Productivity Increase: Production, Utility, Budget Line

$$
\begin{aligned}
c_t^d &= y_t^s - i_t = A_G \left(l_t^d h_t \right)^{\gamma} (k_t)^{1-\gamma} - (g + \delta_k)\, k_t, \\
\frac{c_t^d}{h_t} &= (0.296\,35) \left(l_t^d \right)^{\frac{1}{3}} (1.1578)^{\frac{2}{3}} - (0.02 + 0.05)\,(1.1578).
\end{aligned}
$$

$$
\begin{aligned}
u &= \ln c_t^d + \alpha \ln x_t = \ln c_t + \alpha \ln \left(1 - l_{Ht} - l_t^s \right), \\
-2.644\,8 &= \ln 0.13375 + 1 \ln (0.531) = \ln c_t^d + \ln \left(T_t - l_t^s \right), \\
\frac{c_t^d}{h_t} &= \frac{e^{-2.644\,8}}{(1)\left(1 - 0.185\,26 - l_t^s\right)}.
\end{aligned}
$$

$$
\begin{aligned}
c_t^d &= w_t l_t^s h_t + \rho\,(1 + g)\, k_t^s \\
\frac{c_t^d}{h_t} &= (0.25206)\, l_t^s + 0.052632\,(1 + 0.02)\,(1.1578).
\end{aligned}
$$

Production, Utility Shift up, Higher Wage, Same Labor

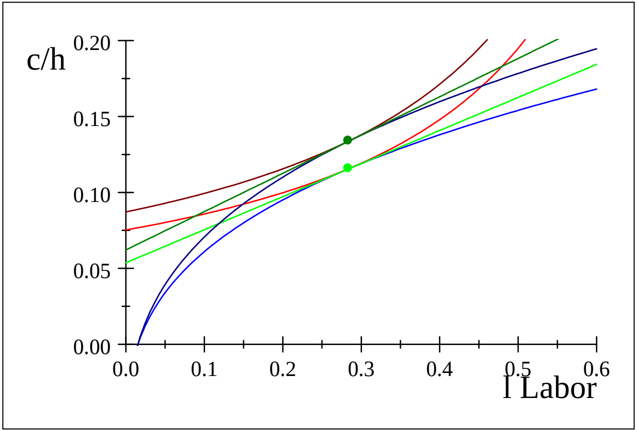


Figure 13.6. General Equilibrium Goods and Labor with Goods Productivity Increase in Example 13.1.

Human Capital Sector Productivity Increase

Example 13.2

- $\alpha = 1$, $\delta_k = 0.05$, $\delta_h = 0.015$, $\beta = \frac{1}{1+\rho} = 0.95$, $A_G = 0.28224$.
- $A_H = 0.189$ increase 5% to $A_H = (0.189)(1.05) = 0.2$.
- $\Longrightarrow g = 0.0333$; $r = 0.1377$, $w = 0.1757$, $l_H = \frac{g+\delta_H}{A_H} = 0.242$, $l = \frac{(00.1377+(1-.05))(1-.95)}{.2} = 0.271\,93$, $x = 0.486$; $\frac{k}{h} = 0.694$.
- Growth rate rises, interest rate rises, wage falls,
  - as human capital rises relative to physical capital.
- Excess Output Demand gives equilibrium g :

$$
\begin{aligned}
Y(g) \quad = \quad & (0.95)\left((0.2) - g + 0.015\right)\left(1 + g + (0.95)(0.05 - 1)\right)\frac{1}{3} \\
& + (0.95)(1+g)(1-(0.95))\frac{2}{3}\left[(0.0526)(1+g) + (g + 0. \right. \\
& -2(1+g)(1-(0.95))\left[1 + g + (0.95)(0.05-1)\right] = 0.
\end{aligned}
$$

Excess Output Demand is Zero at Equilibrium Growth Rate

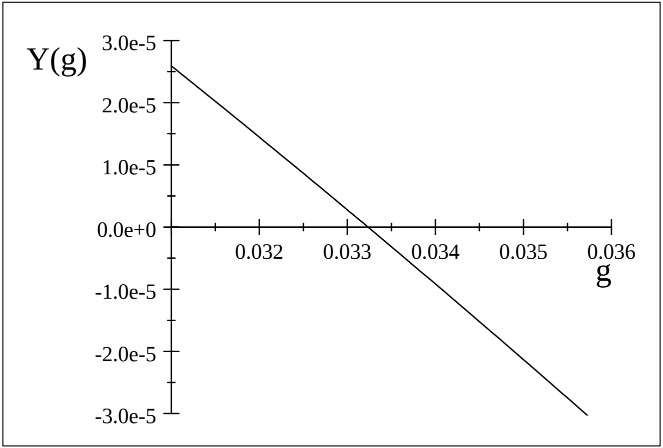


Figure 13.7. Normalized Baseline Excess Demand as a Function of g in Example 13.2.

Analytic Quadratic Solution for Growth

- Write Excess Output Demand as

$$Ag^2 + Bg + C = 0,$$

$$\begin{aligned}
\text{Quadratic term:} \quad & A = -0.3517, \\
\text{Linear term:} \quad & B = -0.0122, \\
\text{Constant term:} \quad & C = 0.00079521. \\
\implies \quad & g = 0.0333.
\end{aligned}$$

$$\frac{1}{w_t} = \frac{1 - \frac{g+\delta_H}{A_H}}{\frac{y_t^d}{h_t}(1+\alpha) - \frac{k_t}{h_t}\left[\rho(1+g) + (g+\delta_k)(1+\alpha)\right]}$$

$$\frac{1}{w_t} = \frac{1}{\gamma A_G}\left(\frac{h_t}{A_G k_t}\right)^{\frac{1-\gamma}{\gamma}}\left(\frac{y_t}{h_t}\right)^{\frac{1-\gamma}{\gamma}};$$

$$\frac{1}{w_t} = \frac{\left(1 - \frac{0.0333+0.015}{0.2}\right)}{y_t^d(1+1) - 0.694\left[(0.0526)(1.0333) + (0.0833)(2)\right]};$$

$$\frac{1}{w_t} = \frac{3}{0.28224}\left(\frac{1}{(0.282\,24)\,0.694}\right)^2 (y_t^s)^2.$$

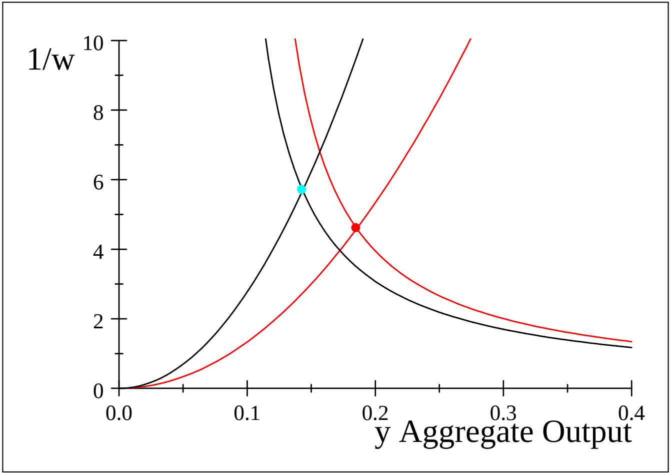


Figure 13.8. $AS - AD$ with Endogenous Growth and a 5% Increase in A_H in Example 13.2.

Example 13.2 Equilibrium Wage

- Excess output demand, with equilibrium growth rate, $\frac{k}{h}$,
  - implies equilibrium wage
  - and can be graphed.

$$
\begin{aligned}
0 &= Y(w_t) = y_t^d - y_t^s \\
& \frac{\left(w_t\left(1-\frac{0.0333+0.015}{0.2}\right)+0.694\left[0.0526\left(1.0333\right)+\left(0.0833\right)\left(2\right)\right]\right)}{1+1} \\
& -0.28224\left(\frac{0.28224}{3w_t}\right)^{0.5}\left(0.694\right)
\end{aligned}
$$

- Wage falls from 0.2177 to 0.1757, a 19.3% decrease.
- When A_H rises, $l_H = \frac{g+\delta_H}{A_H}$ rises as g rises;
  - time endowment for work, leisure falls
  - as $T - l_{Ht}$ goes down, $x + l^s$ goes down.

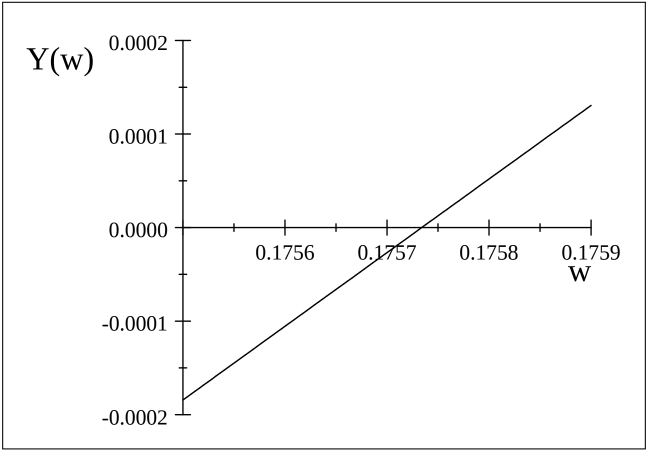


Figure 13.9. Excess Output Demand and the Equilibrium Wage when A_H rises by 5% in Example 13.2.

Consumption and Output

- Consumption, output fall

$$c_t^d = \frac{1}{2}\left[0.1757\left(1-\frac{0.0333+0.015}{0.2}\right)+(0.694)(0.0526)(1.0333)\right]$$
$$= 0.08549,$$

$$y_t^d = 0.1433 =$$
$$\frac{0.1757\left(1-\frac{0.0333+0.015}{0.2}\right)+0.694\left[0.0526(1.0333)+(0.0833)(\right.}{2}$$

- Consumption to output ratio also falls:

$$\frac{c_t^d}{y_t^d} = \frac{0.08549}{0.143\,3} = 0.5966;$$

  - as compared to 0.623 in baseline example.
  - Contrasts to no change when exogenous T changes,
  - with exogenous growth, or when A_G changes.

Example 13.2 Labor Market Effects

- Employment falls 4.3% to 0.27193 when A_H rises 5% :

$$w_t h_t = \frac{\alpha\rho(1+g)(k_t)}{1-(1+\alpha)l_t^s - \frac{(g+\delta_H)}{A_H}};$$

$$w_t h_t = \gamma A_G \frac{\left(\frac{h_t}{k_t}\right)^{\gamma}}{\left(l_t^d\right)^{1-\gamma}} k_t .$$

$$w_t = \frac{(0.694)(0.0526)(1+0.0333)}{1-(1+1)l_t^s - \frac{(0.0333+0.015)}{0.20}};$$

$$w_t = \frac{0.28224}{3} \frac{((0.694))^{\frac{2}{3}}}{\left(l_t^d\right)^{\frac{2}{3}}} .$$

Labor Demand Shifts Back, Labor Supply Shift Out

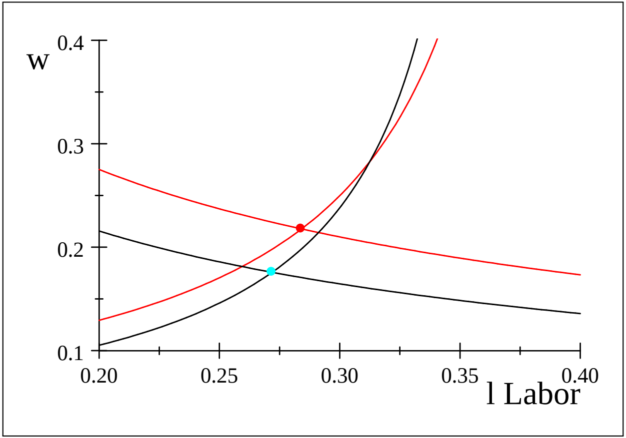


Figure 13.10. Labor Market with Human Capital in Example 13.2.

Isocost, Isoquant and Factor Input Ratio

- Isocost

$$\begin{aligned} 0.1433 &= y_t^s = w_t l_t^d h_t + r_t k_t, \\ \frac{k_t}{h_t} &= \frac{0.1433}{0.1377\,(1)} - \frac{(0.1757)\, l_t^d}{0.1377}. \end{aligned}$$

- Isoquant

$$\begin{aligned} 0.1433 &= y_t^s = A_G \left(l_t^d h_t\right)^{\gamma} (k_t)^{1-\gamma}, \\ \frac{k_t}{h_t} &= \left(\frac{0.1433}{(0.28224)\,(1)\,\left(l_t^d\right)^{\frac{1}{3}}}\right)^{\frac{3}{2}} = \frac{\left(\frac{0.1433}{(0.28224)h_t}\right)^{\frac{3}{2}}}{\left(l_t^d\right)^{\frac{1}{2}}}. \end{aligned}$$

- Factor input ratio

$$\frac{k_t}{h_t l_t^d} = \frac{0.694}{0.27193} = 2.5521.$$

Isocost, Isoquant, Factor Input Ratio, Employment: Down

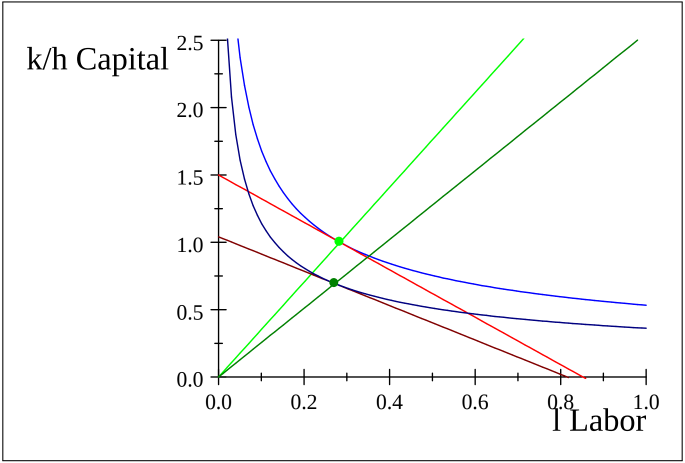


Figure 13.11. Factor Market Equilibrium with Human Capital Productivity Increase in Example 13.2.

Example 13.2 Production, Utility, Budget Line

$$
\begin{aligned}
c_t^d &= y_t^s - i_t = A_G \left(l_t^d h_t \right)^{\gamma} (k_t)^{1-\gamma} - (g + \delta_k) \, k_t , \\
\frac{c_t^d}{h_t} &= (0.28224) \left(l_t^d \right)^{\frac{1}{3}} (0.694)^{\frac{2}{3}} - (0.0333 + 0.05) \, (0.694) .
\end{aligned}
$$

$$
\begin{aligned}
u &= \ln c_t^d + \alpha \ln x_t = \ln c_t + \alpha \ln \left(1 - l_{Ht} - l_t^s \right) , \\
-3.1809 &= \ln 0.08549 + 1 \ln (0.486) , \\
-3.1809 &= \ln c_t^d + \ln \left(T_t - l_t^s \right) , \\
c_t^d &= \frac{e^{-3.1809}}{\left(1 - 0.242 - l_t^s \right)} .
\end{aligned}
$$

$$
\begin{aligned}
c_t^d &= w_t l_t^s h_t + \rho \, (1 + g) \, k_t^s \\
\frac{c_t^d}{h_t} &= (0.1757) \, l_t^s + 0.052632 \, (1 + 0.0333) \, (0.694) .
\end{aligned}
$$

Employment, Wage, Fall as in Contraction

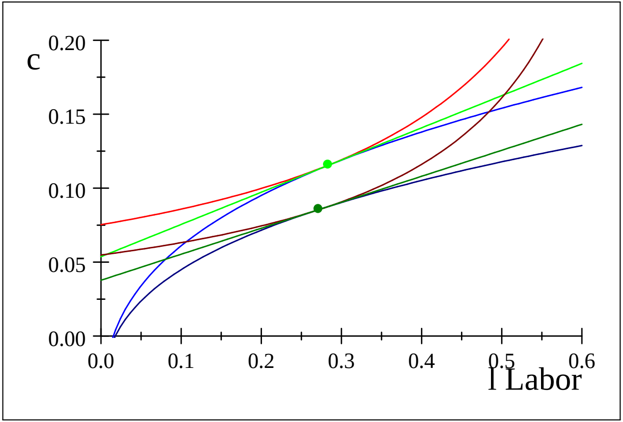


Figure 13.12. General Equilibrium Goods and Labor with Human Capital Productivity Increase in Example 13.2.

Economic Expansion and Contraction

- Change goods, human capital sectoral productivities,
  - A_G, A_H, to explain business cycles,
  - improve upon exogenous growth model.
- Expansion: A_H goes down, l_{Ht}, down, T_t goes up.
  - Rise in T_t, $\frac{k_t}{h_t}$, w_t and l_t, and output $\frac{y_t}{h_t}$.
  - explains salient facts of expansion alone,
  - but can be supplemented by A_G increase.
- Contraction: A_H goes up, employment down.
- Adding A_G change gives bigger changes
  - in wage rate, output level.
  - to make business cycle more pronounced;
  - but no effect on output, consumption/output.

Consumption-Output Puzzle

- Problem: A_{Gt} increase in RBC gives "too-smooth" consumption
  - consumption-output ratio falls by more than data.
  - Known as consumption-output puzzle.
  - Stochastic rise in A_{Gt} causes $\frac{c}{y}$ to fall.
- In our model, expansion with A_G increase:
  - no change in $\frac{c}{y}$,
  - while decrease in A_H :
  - causes consumption-output ratio to rise.
- Made stochastic, with A_H decrease, A_G increase,
  - then consumption-output falls by less,
  - more like the data.
- Similar solution to consumption-output puzzle
  - in 2000 Maffezzoli article in international context.

Two Sector Business Cycle Models

- Hansen 1985, Rogerson 1988:
  - modify standard exogenous growth *RBC*,
  - add external labor margin to enter labor force,
  - explain changes in employment,
  - similar to our changes in T
- Greenwood, Hercowitz 1991, Benhabib et al 1991
  - second sector called non-market, household, sector.
  - Labor: goods to household sector in expansion;
  - household to goods sector in contraction.
  - Productivities change in each sector.
- Here: second sector is also a household sector:
  - produces human capital investment in particular.
  - Can have shocks to such 2-sector models:
  - as Jones et al. 2005: goods, human capital sectors.

Cyclical Change in Growth Rate

- A_H falls, expansion, growth rate g falls.
  - counter-intuitive: expansions have higher output.
  - Business cycle facts show small correlation
  - between growth rate and output level.
- Output growth rate starts falling early in expansions
  - starts rising just as contractions are starting
  - Fall in A_H, in growth rate, consistent with expansions.
- Steady state comparative static explanation
  - simply a first approximation,
  - with A_G and A_H being stochastic more generally.
- Stochastic shocs to A_G, A_H gradually rise and fall.
  - Process abstracted from
  - with shifts in productivity parameters.
- Stochastic shocks:.better growth rate cycle correlation

Shocks within a Business Cycle Model

- Let output have a shock factor Z_t :

$$y_t = A_{Gt} Z_t \left(l_t\right)^{\gamma} \left(k_t\right)^{1-\gamma}.$$

- Assume "log-normally distributed", so let

$$e^{z_t} \equiv Z_t, \tag{1}$$

$$z_t = \phi z_{t-1} + \varepsilon_t, \tag{2}$$

  - ϕ is persistence factor, between 0 and 1,
  - usually close to one, so z_t changes slowly.
  - ε_t normally distributed, mean zero.

- Implies that $\ln y_t$ normally distributed,
  - y_t "log-normally distributed",
  - as consistent with macroeconomic data.

Example 13.3. Zero Persistence

- Let $z_0 = 0. \implies Z_0 = e^{z_0} = e^0 = 1.$
- Assume this shock sequence:

$$\begin{aligned}
\varepsilon_1 &= 0.1; \\
\varepsilon_2 &= 0.1; \\
\varepsilon_3 &= 0.0; \\
\varepsilon_4 &= -0.1.
\end{aligned}$$

- Zero persistence $\implies \phi = 0$ and $z_t = \varepsilon_t$:

$$\begin{aligned}
z_1 &= \phi z_0 + \varepsilon_1 = (0)\,(0) + 0.1 = 0.1; \\
&\implies z_1 = \varepsilon_1;\ z_2 = \varepsilon_2;\ z_3 = \varepsilon_3;\ z_4 = \varepsilon_4; \\
Z_1 &= e^{\varepsilon_1} = e^{0.1} = 1.1, \\
Z_2 &= e^{\varepsilon_2} = e^{0.1} = 1.1, \\
Z_3 &= e^{\varepsilon_3} = e^{0} = 1.0, \\
Z_4 &= e^{\varepsilon_4} = e^{-0.1} = 0.90.
\end{aligned}$$

Example 13.4. Full Persistence

- Let $z_0 = 0$, $\phi = 1$: Gives more gradual rise, fall:

$$
\begin{aligned}
z_1 &= \phi z_0 + \varepsilon_1 = (1)(0) + 0.1 = 0.1, \\
&\Longrightarrow Z_1 = e^{z_1} = e^{0.1} = 1.1; \\
z_2 &= z_1 + \varepsilon_2 = 0.1 + 0.1 = 0.2, \\
&\Longrightarrow Z_2 = e^{z_2} = e^{0.2} = 1.22; \\
z_3 &= z_2 + \varepsilon_3 = 0.2 + 0 = 0.2, \\
&\Longrightarrow Z_3 = e^{z_3} = e^{0.2} = 1.22; \\
z_4 &= z_3 + \varepsilon_4 = 0.2 - 0.1 = 0.1, \\
&\Longrightarrow Z_4 = e^{z_4} = e^{0.1} = 1.1.
\end{aligned}
$$

- $Z_0 = 1; Z_1 = 1.1; Z_2 = 1.22; Z_3 = 1.22; Z_4 = 1.1.$
    - More gradual rise, fall, in Z more data consistent.
    - Same as series of comparative static changes.

Example 13.5. High Degree of Shock Persistence

- Typical assume near to $\phi = 0.9$:

$$
\begin{aligned}
z_0 &= 0; \\
z_1 &= \phi z_0 + \varepsilon_1 = (0.9)(0) + 0.1 = 0.1; \\
z_2 &= (0.9) z_1 + \varepsilon_2 = 0.9(0.1) + 0.1 = 0.19; \\
z_3 &= (0.9) z_2 + \varepsilon_3 = 0.9(0.19) + 0 = 0.17; \\
z_4 &= (0.9) z_3 + \varepsilon_4 = 0.9(0.17) - 0.1 = 0.05.
\end{aligned}
$$

$$
\begin{aligned}
Z_1 &= e^{0.1} = 1.1, \\
Z_2 &= e^{0.19} = 1.21, \\
Z_3 &= e^{0.17} = 1.19, \\
Z_4 &= e^{0.05} = 1.05.
\end{aligned}
$$

- Not so far from comparative static changes.

Growth Accounting

- A_{Gt} explains exogenous growth residuals,
  - after accounting for labor, capital contribution.
  - Called "growth accounting".
- Different result: on BGP within endogenous growth
-

$$\begin{aligned} Endogenous &: \quad y_t = A_G h_t^{\gamma} (l_t)^{\gamma} (k_t)^{1-\gamma}, \\ Exogenous &: \quad y_t = A_{Gt} (l_t)^{\gamma} (k_t)^{1-\gamma}, \\ Link &: \quad A_{Gt} = A_G h_t^{\gamma}. \end{aligned}$$

  - Time change in A_{Gt} from h_t in endogenous growth.

Exogenous Growth Accounting

$$\frac{y_{t+1}}{y_t} = \frac{A_{Gt+1}\left(l_{t+1}\right)^{\gamma}\left(k_{t+1}\right)^{1-\gamma}}{A_{Gt}\left(l_t\right)^{\gamma}\left(k_t\right)^{1-\gamma}}.$$

- Let g_y, g_A, g_l, g_k be growth rate of y_t, A_{Gt}, l_t, k_t.

$$\begin{aligned} 1+g_y &= \left(1+g_A\right)\left(1+g_l\right)^{\gamma}\left(1+g_k\right)^{1-\gamma}, \\ g_y &\simeq g_A + \gamma g_l + \left(1-\gamma\right) g_k;\ g_l = 0; \\ &\Longrightarrow g_A = g_y - \left(1-\gamma\right) g_k. \end{aligned}$$

- Example: $1-\gamma = 0.4$, $g_y = 0.02$, $g_k = 0.01$,
  - $g_A = 0.02 - 0.4\left(0.01\right) = 0.016$, or 1.6% from *TFP*.
  - From A_{Gt} : large; suggests exogenous *TFP* important.

Endogenous Growth Accounting

- $$\begin{aligned} \frac{y_{t+1}}{y_t} &= \frac{A_G (h_{t+1})^{\gamma} (l_{t+1})^{\gamma} (k_{t+1})^{1-\gamma}}{A_G (h_t)^{\gamma} (l_t)^{\gamma} (k_t)^{1-\gamma}}; \\ g_y &\simeq \gamma g_h + \gamma g_l + (1-\gamma) g_k; \; g_l = 0; \\ g_y - (1-\gamma) g_k &= \gamma g_h. \end{aligned}$$
- Same measure of TFP $[g_y - (1-\gamma) g_k]$, now given by γg_h.
  - Growth rate of h_t, factored by γ, is exactly TFP residual.
  - qualifies TFP literature: could simply be measure of h_t.
  - Along BGP, it is an identity that γg_h explains TFP measure:

$$\begin{aligned} g_y - (1-\gamma) g_k &= \gamma g_h; \\ 0.020 - \left(\frac{2}{3}\right) 0.020 &= \frac{1}{3} 0.020. \end{aligned}$$

  - Long run TFP issue may be measurement problem.
- Emphasizes: A_H change may be as important as A_G change.

Historical Trends with Endogenous Growth

- A_G change does not effect endogenous growth, labor hours.
  - cannot explain historical trend up in growth rate,
  - or trend down in labor hours.
- Very small trend upwards in A_H can explain
  - growth trend,
  - time allocation trends,
  - infrastructure trends,
  - trends in historical evolution of industry

Time Allocation Trends

- Fall in exogenous T cannot directly explain education time.
- Small trend up in A_H gives trend up in l_H
  - in Example 12.1;
  - this explains historical trend up in education time:
  - from elementary school level, to high school level,
  - to university level, and now to graduate degree levels.
- Also can explain trend down in labor hours per week:
  - as education time increases, time leftover decreases,
  - endogenous T decreases and working time decreases.

- Growth rate rise: from zero per-capita income growth in Europe,
  - known as Malthusian 1798 zero growth equilibrium,
  - to higher growth rate during industrial revolution, now.
  - Explained also by small gradual rise in A_H.
- Lucas 2002 explains transformation from
  - low growth Malthusian to high growth industrial revolution
  - with investment in human capital passing some critical level.
  - Demographic changes occurs as greater investment
  - "quality" of children, education, than in quantity.
  - Goes back to Becker, Lewis 1973, Becker, Barro 1988.

- No accepted explanation why A_H might be rising.
  - Might be whole cultural technology advance,
  - or "spillovers" of human, physical capital.
  - Example: policy of building schools in China under Deng:
  - cause A_H there to rise over time?
- "Why Doesn't Capital Flow from Rich to Poor Countries?"
  - Lucas 1990: cause insufficient human capital;
  - physical to human capital ratio has an equilibrium.
  - Goes back to T.W. Schultz 1965:
  - moving from traditional to moder agriculture.

Industrial Transformation

- "Structural transformation": sectoral relative shifts
  - agriculture to manufacturing to services,
  - as economies develop.
- No simple theory to explain with exogenous growth.
- Human capital endogenous growth theory
  - gives very simple way to explain:
  - based on different degrees of human capital usage,
  - combined with slight trend upwards in A_H

Mathematical Theory of Structural Transformation

- agriculture output y_a, manufacturing y_m services y_s :

$$\begin{aligned} y_{at} &= A_a \left(l_{at} h_t\right)^{\gamma_a} \left(s_{at} k_t\right)^{1-\gamma_a} ; \\ y_{mt} &= A_m \left(l_{mt} h_t\right)^{\gamma_m} \left(s_{mt} k_t\right)^{1-\gamma_m} ; \\ y_{st} &= A_s \left(l_{st} h_t\right)^{\gamma_s} \left(s_{st} k_t\right)^{1-\gamma_s} ; \\ i_{Ht} &= A_H \left(l_{Ht} h_t\right)^{\eta} \left(s_{Ht} k_t\right)^{1-\eta} . \end{aligned}$$

  - Shares of time, capital constraint add to one:

$$s_{at} + s_{mt} + s_{st} + s_{Ht} = 1.$$

$$l_{at} + l_{mt} + l_{st} + l_{Ht} + x_t = 1.$$

- Human capital intensity: agriculture < manufacturing < services

$$\frac{w_t l_{at} h_t}{y_{at}} = \gamma_a < \frac{w_t l_{mt} h_t}{y_{mt}} = \gamma_m < \frac{w_t l_{st} h_t}{y_{st}} = \gamma_s .$$

  - Land value makes agriculture low in human capital intensity.
  - Services highest in human capital intensity: example: finance.
- As A_H trends up, human capital intensive sectors expand

Policy

- Important to raise A_H.
  - by infrastructure development generally,
  - especially schools, hospitals, health education, service.
- Can also raise physical capital investment productivity.
  - through an i_{Kt} function, with parameter A_{Kt},
  - could be raised through physical capital infrastructure.
- For example, cost of physical capital investment:

$$i_{Kt} = A_K \left(l_{Kt} h_t\right)^{\kappa} \left(s_{Kt} k_t\right)^{1-\kappa},$$

  - with l_{Kt} and s_{Kt} labor, capital shares.
  - Raising A_K, just as raising A_H, could increase growth.
- Would be extension of baseline models,
  - modeling investment i_t with separate sector,
  - with technology $A_K \left(l_{Kt} h_t\right)^{\kappa} \left(s_{Kt} k_t\right)^{1-\kappa}$,
  - instead of output turned costlessly into investment,
  - with $i_t = y_t - c_t$.
- Broadly could focus on raising both A_H, A_K.

Application: Capital Symmetry in Growth Theory

- Stigler 1939: flexible output scale, higher marginal cost.
- Capacity utilization rate less than one here.
  - used for better explaining business cycles.
  - Can also be used for a symmetry growth theory,
  - regarding human versus physical capital.
- $r_t - \delta_k = A_H (1 - x_t) - \delta_H$: asymmetric matching,
  - of variable marginal product of capital, r_t,
  - to utilization rate of human capital, $1 - x_t$.
- Utilization rate u_t, then $1 - u_t$ spare capacity.
  - and assume unused capital does not depreciate:

$$\begin{aligned} y_t &= A_G (l_t h_t)^{\gamma} (u_t k_t)^{1-\gamma}, \\ i_t &= k_{t+1} - k_t + \delta_k u_t k_t. \end{aligned}$$

- Friedman 1976 "entrepreneurial capacity"
  - enters utility as $\psi \ln (1 - u_t)$, $\psi \geq 0$,
  - so get utility when not full capital utilization,
  - just as leisure enters utility function.

Capital Symmetry Growth Model 1

$$c_t = A_G (l_t h_t)^{\gamma} (u_t k_t)^{1-\gamma} - k_{t+1} + k_t - \delta_k u_t k_t,$$

$$\begin{aligned} V(k_t, h_t) &= \underset{l_{Ht,} x_t, u_t, k_{t+1}, h_{t+1}}{Max} \\ & \ln \left[A_G \left[(1 - x_t - l_{Ht}) h_t \right]^{\gamma} (u_t k_t)^{1-\gamma} - k_{t+1} + k_t - \delta_k u_t k_t \right] \\ & + \ln x_t + \psi \ln (1 - u_t) + \beta V(k_{t+1}, h_{t+1}). \end{aligned}$$

$$\begin{aligned} BGP &: \quad 1 + g_t = \frac{1 + (r_t - \delta_k) u_t}{1 + \rho} = \frac{1 + A_H (1 - x_t) - \delta_H}{1 + \rho}, \\ r_t &= (1 - \gamma) A_G \left[\frac{l_t h_t}{u_t k_t} \right]^{\gamma}. \end{aligned}$$

Capital Symmetry Growth Model 2

Fully Symmetric between Human, Physical Capital

- Add:
  - Unused human capital also does not depreciate:

$$h_{t+1} = h_t - \delta_h \left(1 - x_t\right) h_t + i_{Ht}.$$

  - Let physical capital enter human capital investment:

$$\begin{aligned} s_{Gt} + s_{Ht} &= 1, \\ y_t &= A_G \left(l_t h_t\right)^{\gamma} \left(s_{Gt} u_t k_t\right)^{1-\gamma}; \\ i_{Ht} &= A_H \left(l_{Ht} h_t\right)^{\eta} \left(s_{Ht} u_t k_t\right)^{1-\eta}. \end{aligned}$$

- Get: equal returns on human, physical capital, again,
- But now each return consists of two parts:
  - marginal product of capital
  - capacity utilitization rate of capital
- And depreciation rate enters symmetrically.

$$V\left(k_t, h_t\right) = \underset{l_{Ht,}x_t, u_t, k_{t+1}, h_{t+1}}{Max}$$

$$\ln\left[A_G\left[\left(1 - x_t - l_{Ht}\right)h_t\right]^{\gamma}\left[\left(1 - s_{Ht}\right)u_t k_t\right]^{1-\gamma} - k_{t+1} + k_t - \delta_k u_t k_t\right]$$
$$+ \ln\left(x_t\right) + \beta V\left(k_{t+1}, h_t - \delta_h\left(1 - x_t\right)h_t + A_H\left(l_{Ht} h_t\right)^{\eta}\left(s_{Ht} u_t k_t\right)^{1-\eta}\right);$$

$$g_t = \frac{\left(r_t - \delta_k\right)u_t - \rho}{1 + \rho} = \frac{\left(r_{Ht} - \delta_h\right)\left(1 - x_t\right) - \rho}{1 + \rho},$$

$$r_{Ht} \equiv \eta A_H\left(\frac{l_{Ht} h_t}{s_{Ht} u_t k_t}\right)^{\eta-1}.$$

$$\left(r_t - \delta_k\right)u_t = \left(r_{Ht} - \delta_h\right)\left(1 - x_t\right).$$