# Advanced Modern Macroeconomics

## Savings and Investment

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- Goods output divided between consumption and savings.
- Consumer savings goes to firms which use it for investment.
- Consumer subtracts capital from consumption today
    - to yield more consumption in next period.
    - this defines the capital available to firms for production.
- Makes for capital supply and capital demand.
- Only a two period model; initial output given exogenously.
- All capital used up in production process.
- Intertemporal problem with consumption smoothing across time.

# Building on the Last Chapters

- Last chapter on intratemporal substitution: goods, leisure.
- Same tools now for capital theory, intertemporal substitution.
- Instead of goods-leisure choice: consumption today, tomorrow.
- Mathematical form of budget almost identical Part 2.
  - Allocation of time constraint, with given time endowment.
  - now allocation of consumption, investment with given goods endowment.
  - Budget constraint also similar when decentralized:
    - consumption equals labor wages plus profits,
    - now consumption equals capital savings return plus profit.
- Decentralization: marginal product of capital key instead of marginal product of labor.

# Learning Objective

- The student conceptualizes how capital is saved and invested
- Key step of intertemporal consumption smoothing,
- Focus on intertemporal margin in simplest framework
- Sets up understanding margin in full dynamic economy.
- See margin as key to determining growth rate of the economy.

# Who Made It Happen: Capital Theory

- Adam Smith 1776: output equals current consumption and capital
  - "*The general stock of any country ... naturally divides itself into... that portion which is reserved for immediate consumption, and ... fixed capital...and circulating capital... that... affords a revenue or profit.*
- Bohm-Bawerk's 1888 *The Positive Theory of Capital*
- Fisher: 2-period general equilibrium intertemporal optimization,
  - 1896 *Appreciation and Interest*, 1907 *The Rate of Interest*, 1930 *Theory of Interest*,
- Ramsey's 1928 "A Mathematical Theory of Saving"
  - modern general equilibrium capital accumulation over all time.
- Knut Wicksell's 1901 *Lectures on Political Economy*
  - capital a factor used in production of output, along with only labor.
  - No more classic theory of land, labor, capital as inputs
- Friedman's 1957 *Theory of the Consumption Function*
  - permanent income hypothesis of consumption, 2-period model.
- Solow 1956 explains economic growth.

# Representative Agent Intertemporal Economy

$$\underset{c_0, c_1, k_1, y_1}{\text{Max}} \; u\left(c_0, c_1\right)$$

$$y_0 = c_0 + k_1.$$

$$c_1 = y_1 = f\left(k_1\right).$$

$$\underset{k_1}{\text{Max}} \; u\left[y_0 - k_1, f\left(k_1\right)\right].$$

$$\frac{\partial u\left(c_0, c_1\right)}{\partial c_0}\frac{\partial\left(y_0 - k_1\right)}{\partial k_1} + \frac{\partial u\left(c_0, c_1\right)}{\partial c_1}\frac{\partial\left[f\left(k_1\right)\right]}{\partial k_1} = 0.$$

$$MP_k \equiv \frac{\partial f\left(k_1\right)}{\partial k_1} = \frac{\frac{\partial u(c_0, c_1)}{\partial c_0}}{\frac{\partial u(c_0, c_1)}{\partial c_1}} \equiv MRS_{c_0, c_1}$$

# Log-utility and Cobb-Douglas Production

$$u = \ln c_0 + \beta \ln c_1,$$

- $\beta \in (0, 1)$ parameter of "time preference".

$$MRS_{c_0,c_1} \equiv \frac{\frac{\partial \ln c_0}{\partial c_0}}{\beta \frac{\partial \ln c_1}{\partial c_1}} = \frac{\frac{1}{c_0}}{\frac{\beta}{c_1}},$$

$$y_t = A_G l_t^{\gamma} k_t^{1-\gamma},$$

- $\gamma \in (0, 1)$ , $l_t = 1$, production occurs only at time $t = 1$,

$$y_1 = A_G k_1^{1-\gamma},$$

$$A_G (1-\gamma) k_1^{-\gamma} = \frac{\partial f}{\partial k_1} = \frac{\frac{\partial u(c_0,c_1)}{\partial c_0}}{\beta \frac{\partial u(c_0,c_1)}{\partial c_1}} = \frac{\frac{1}{c_0}}{\frac{\beta}{c_1}}, \tag{1}$$

# Example 5.1

- Target a reasonable growth rate in consumption of 2.5%.
- Assume $\beta = 0.98$, $\gamma = 0.5$, $A_G = 12$, $y_0 = 100$ .

$$u = \ln c_0 + (0.98) \ln c_1, \tag{2}$$

$$c_1 = y_1 = 12\sqrt{k_1}.$$

$$100 = c_0 + k_1.$$

$$\underset{k_1}{\mathrm{Max}} u(c_0, c_1) = \ln(100 - k_1) + (0.98) \ln\left(12\sqrt{k_1}\right).$$

$$-\frac{1}{100 - k_1} + \frac{(0.98)(0.5)12k_1^{-0.5}}{12k_1^{0.5}} = 0. \tag{3}$$

# Equilibrium Solution

$$k_1 = \frac{(98)(0.5)}{1 + (.98)(0.5)} = 32.886. \tag{4}$$

$$c_1 = 12\sqrt{32.886} = 68.816, \tag{5}$$

$$c_0 = 100 - 32.886 = 67.114. \tag{6}$$

$$1 + g \equiv \frac{c_1}{c_0} = \frac{68.82}{67.11} = 1.0255,$$

- $g = 0.0255$, or about 2.55%.

$$u = \ln 67.114 + (0.98) \ln (68.816) = 8.3532. \tag{7}$$

$$c_1 = \left(\frac{e^u}{c_0}\right)^{\frac{1}{\beta}} = \left(\frac{e^{8.3532}}{c_0}\right)^{\frac{1}{0.98}}. \tag{8}$$

$$c_1 = 12\sqrt{k_1} = 12\sqrt{100 - c_0}. \tag{9}$$

# Graphically

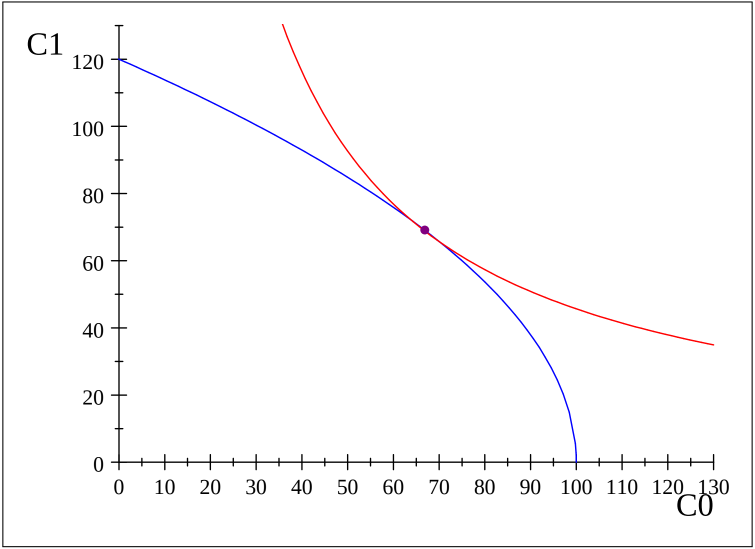


Figure 5.1. Intertemporal Consumption and Investment in Example 5.1.

# Productivity Change: Example 5.2

$\beta = 0.98$, $\gamma = 0.5$, and $A_G$ doubles to 24 from 12 : $c_1 = y_1 = 24\sqrt{k_1}$.

$$\underset{k_1}{\text{Max}} u = \ln(100 - k_1) + (0.98)\ln\left(24\sqrt{k_1}\right).$$

$$-\frac{1}{100 - k_1} + \frac{(0.98)(0.5)24k_1^{-0.5}}{24k_1^{0.5}} = 0. \tag{10}$$

Same solution for capital $k_1$:

$$k_1 = \frac{(98)(0.5)}{1 + (.98)(0.5)} = 32.89.$$

But next period consumption doubles:

$$c_1 = 24\sqrt{32.89} = 137.64.$$

Current consumption the same

$$c_0 = 100 - 32.89 = 67.11.$$

Utility rises

$$u = \ln 67.11 + (0.98)\ln(137.64) = 9.03.$$

# Productivity Doubles

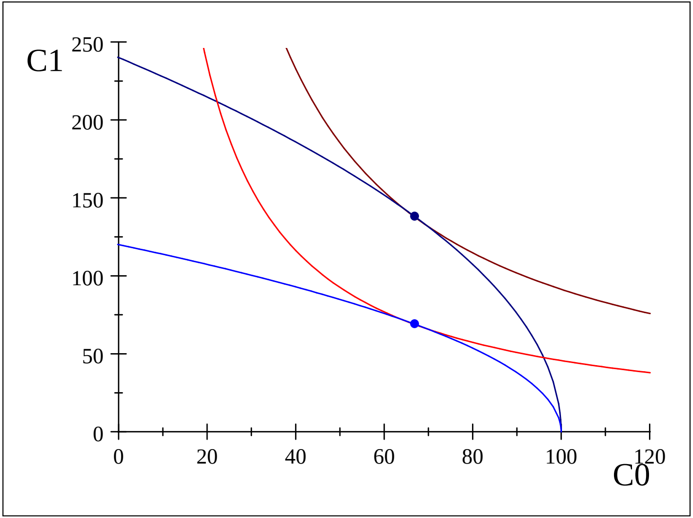


Figure 5.2. Intertemporal Consumption with Doubling of Productivity in Example 5.2.

# Offsetting Income and Substitution Effects

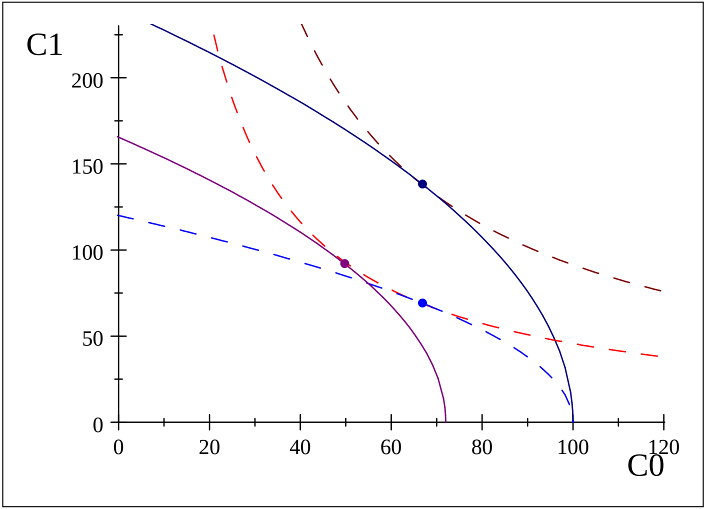


Figure 5.3. Substitution Effect from Doubling of Capital Productivity in Example 5.2.

# Decentralized Consumer and Firm Problems

- Lend out $k_1$ this period, get $(1+r)\,k_1$ next period: capital plus interest.
- Market clearing

$$k_1^s = k_1^d, \tag{11}$$

$$c_1^d = c_1^s = y_1.$$

$$u = \ln c_0^d + \beta \ln c_1^d,$$

$$c_0^d = y_0 - k_1^s. \tag{12}$$

$$c_1^d = \Pi_1 + k_1^s\,(1+r)\,. \tag{13}$$

# Wealth Constraint Form of Budget Constraints

- $$\frac{c_1^d}{1+r} = \frac{\Pi_1}{1+r} + k_1^s;$$
$$k_1^s = y_0 - c_0^d;$$
$$c_0^d + \frac{c_1^d}{1+r} = y_0 + \frac{\Pi_1}{1+r}.$$

- With $\Pi_1$ next period income, wealth $W$ :

$$W \equiv y_0 + \frac{\Pi_1}{1+r} = c_0^d + \frac{c_1^d}{1+r}. \tag{14}$$

- Present discounted value of income equals discounted value of consumption
- Use either wealth constraint or two budget constraints

# Consumer Problem

$$\underset{c_0, c_1, k_1^s}{\text{Max}}\ u = \ln c_0^d + \beta \ln c_1^d$$

$$c_0^d = y_0 - k_1^s, \tag{15}$$

$$c_1^d = \Pi_1 + k_1^s\,(1+r)\,. \tag{16}$$

$$\underset{k_1^s}{\text{Max}}\ u = \ln\left(y_0 - k_1^s\right) + \beta \ln\left[\Pi_1 + (1+r)\,k_1^s\right].$$

$$\frac{\beta\,(1+r)}{\Pi_1 + k_1^s(1+r)} - \left(\frac{1}{y_0 - k_1^s}\right) = 0.$$

$$k_1^s = \frac{y_0\beta}{1+\beta} - \frac{\Pi_1}{(1+\beta)\,(1+r)}. \tag{17}$$

# Firm Problem

$$\underset{k_1^d}{\text{Max}}\ \Pi_1 = y_1 - (1+r)\,k_1^d,$$

$$y_1 = f(k_1^d) = A_G\left(k_1^d\right)^{1-\gamma}.$$

$$\underset{k_1^d}{\text{Max}}\ \Pi_1 = A_G\left(k_1^d\right)^{1-\gamma} - (1+r)\,k_1^d. \tag{18}$$

$$\frac{\partial f(k_1^d)}{\partial k_1^d} = (1-\gamma)\,A_G\left(k_1^d\right)^{-\gamma} = 1+r. \tag{19}$$

# Example 5.3: Firm

$A_G = 12,\ \gamma = 0.5,\ y_0 = 100,\ \beta = 0.98:$

$$y_1 = 12\left(k_1^d\right)^{0.5};$$

$$(0.5)\,12\left(k_1^d\right)^{-0.5} = 1 + r. \tag{20}$$

$$k_1^d = \frac{36}{(1+r)^2}. \tag{21}$$

$$\Pi_1 = 12\sqrt{\frac{36}{(1+r)^2}} - \frac{36\,(1+r)}{(1+r)^2} = \frac{36}{1+r}. \tag{22}$$

$$y_1 = 12\sqrt{\frac{36}{(1+r)^2}} = \frac{72}{1+r}. \tag{23}$$

# Example 5.3: Capital Market Equilibrium

$$k_1^s = \frac{y_0\beta}{1+\beta} - \frac{\Pi_1}{(1+\beta)(1+r)} = \frac{y_0\beta}{1+\beta} - \left(\frac{1}{1+\beta}\right)\frac{36}{(1+r)^2}. \tag{24}$$

$$k_1^d = \frac{36}{(1+r)^2} = \frac{y_0\beta}{1+\beta} - \left(\frac{1}{1+\beta}\right)\frac{36}{(1+r)^2} = k_1^s. \tag{25}$$

$$r = \sqrt{\frac{36(2+\beta)}{y_0\beta}} - 1 = \sqrt{\frac{36(2.98)}{98}} - 1 = 0.0463. \tag{26}$$

$$k_1^d = \frac{36}{(1+r)^2} = \frac{36}{(1.0463)^2} = 32.89;$$

$$k_1^s = \frac{98}{1.98} - \left(\frac{1}{1.98}\right)\frac{36}{(1.0463)^2} = 32.89.$$

$$1+r = \frac{6}{\left(k_1^d\right)^{0.5}},\ 1+r = \frac{6}{\left(y_0\beta - (1+\beta)\,k_1^s\right)^{0.5}} \tag{27}$$

# Example 5.3: Rest of Equilibrium

$$y_1 = \frac{72}{1+r} = \frac{72}{1.0463} = 68.81. \quad (28)$$

$$\Pi_1 = \frac{36}{1+r} = \frac{36}{1.0463} = 34.41. \quad (29)$$

$$c_0^d = 100 - k_1^s = 67.12. \quad (30)$$

$$y_1 = c_1^d = \Pi_1 + k_1^s (1+r) = 34.41 + 32.88 (1.0463) = 68.81. \quad (31)$$

$$1 + g = \frac{c_1}{c_0} = \frac{68.82}{67.11} = 1.0255,$$

$$u = \ln 67.12 + 0.98 \ln 68.81 = 8.35.$$

- Savings rate: $\frac{k_1}{y_0} = \frac{32.88}{100}$, or 33%.

# Example 5.3: Capital Market Graph

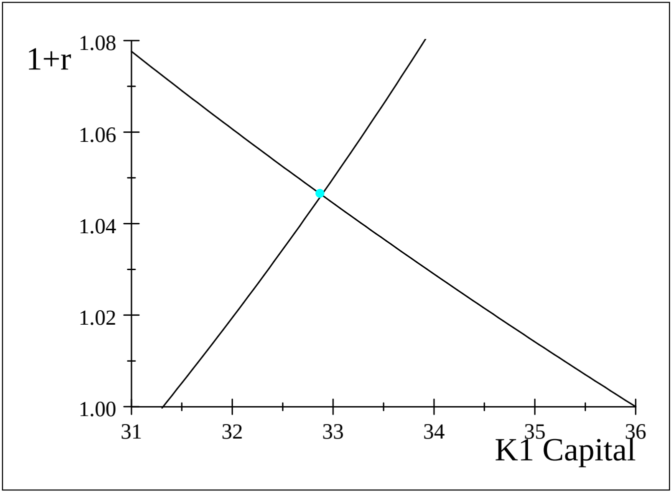


Figure 5.4. Savings and Investment: Aggregate Supply and Demand for Capital in Example 5.3.

$$c_0 = y_0 - k_1^s = 50.51 + \frac{18.18}{(1+r)^2}. \tag{32}$$

$$c_1^d = \Pi_1 + k_1^s (1+r) = 49.49\,(1+r) + \frac{17.82}{1+r}.$$

$$c_1^s = y_1 = \frac{72}{1+r}. \tag{33}$$

$$u = \ln 67.12 + (0.98) \ln 68.81 = 8.35.$$

$$\frac{1}{1+r} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{c_1^d \pm \sqrt{\left(c_1^d\right)^2 - 4\,(17.82)\,(49.49)}}{2\,(17.82)}. \tag{34}$$

$$\frac{1}{1+r} = \frac{c_1^s}{72}. \tag{35}$$

# Future Goods Market Graphically

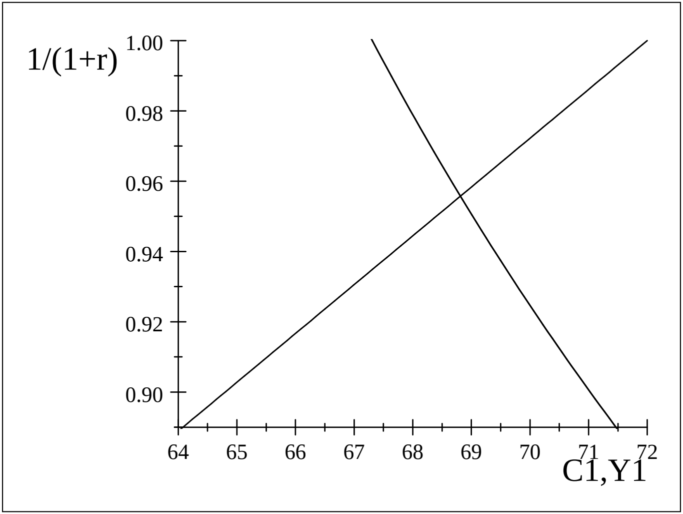


Figure 5.5. Market for Future Period Consumption in Example 5.3.

# General Equilibrium: Add Budget Line

$$c_1^d = \Pi_1 + k_1^s(1+r) = \frac{36}{1.0463} + (100 - c_0)(1.0463).$$

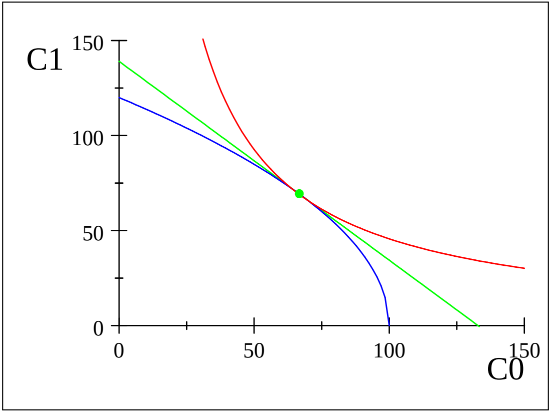


Figure 5.6. General Equilibrium Intertemporal Consumption with Market Line in Example 5.3.

# Example 5.4: Productivity Increase by 5%

$\gamma = 0.5,\ y_0 = 100,\ \beta = 0.98;\ A_G$ : up by 5% to $12\,(1.05) = 12.6$

$$\underset{k_1^d}{\text{Max}}\ \Pi_1 = 12.6\sqrt{k_1^d} - (1+r)k_1^d, \tag{36}$$

$$k_1^d = \frac{39.69}{(1+r)^2}, \Pi_1 = \frac{39.69}{1+r}.y_1 = \frac{79.38}{1+r}. \tag{37}$$

$$1 + r = \sqrt{\frac{39.69}{k_1^d}}. \tag{38}$$

$$k_1^s = 49.49 - \frac{20.045}{(1+r)^2},\ 1 + r = \sqrt{\frac{20.045}{49.49 - k_1^s}}. \tag{39}$$

$$k_1^s = 49.49 - \frac{20.045}{(1+r)^2} = \frac{39.69}{(1+r)^2} = k_1^d.$$

$$1 + r = \sqrt{\frac{(39.69 + 20.045)}{49.49}} = 1.0986,$$

# Graphical Capital Market: No Change in Capital

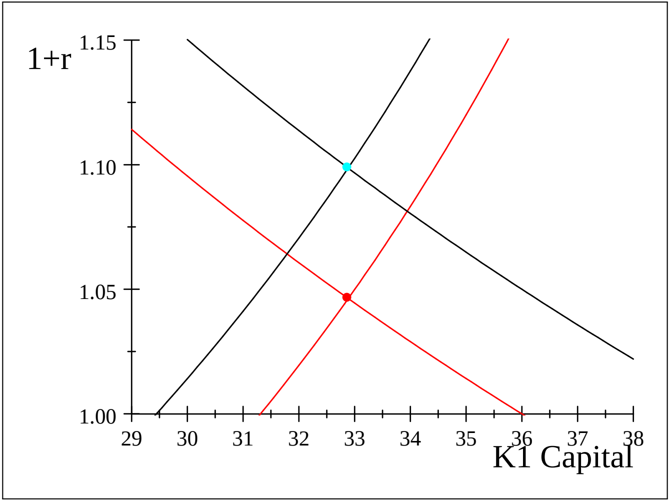


Figure 5.7. Shift in Supply and Demand for Capital from a Productivity Increase in Example 5.4.

# Future Goods Market

$$c_1^d = 49.49\left(1+r\right) + \frac{19.645}{1+r}. \tag{40}$$

$$\frac{1}{1+r} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{c_1^d \pm \sqrt{\left(c_1^d\right)^2 - 4\left(19.6\right)\left(49.5\right)}}{2\left(19.645\right)}. \tag{41}$$

$$\begin{aligned} c_1^s &= \frac{79.38}{1+r}, \\ \frac{1}{1+r} &= \frac{c_1^s}{79.38}. \end{aligned} \tag{42}$$

# Future Goods Market Graphically: Output Rises

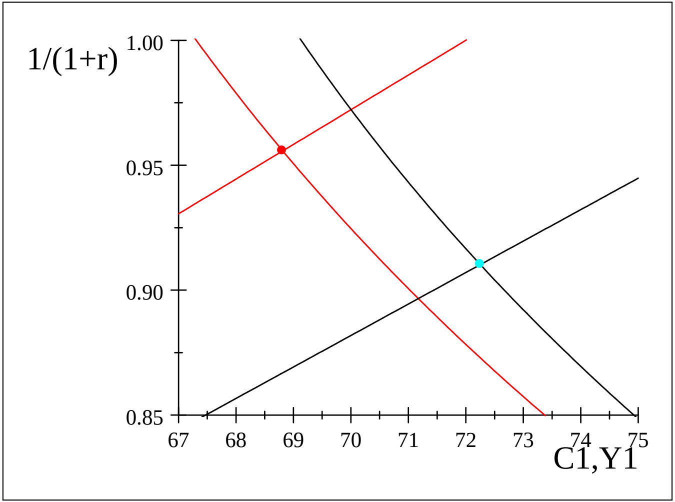


Figure 5.8. Market for Future Period Consumption with 5% Productivity Increase in Example 5.4 (black) versus Example 5.3 (red).

# Current Period Income Endowment: Example 5.5

- $y_0$ rise by 5% to 105, all other parameters at baseline.

$$k_1 = \frac{105\,(0.98)\,(0.5)}{1 + (0.98)\,(0.5)} = 34.53.$$

- Savings rate the same:

$$\frac{k_1}{y_0} = \frac{34.53}{105} = 0.3289.$$

- But capital investment rises.

$$c_0 = 105 - 34.53 = 70.47.$$

$$c_1^s = y_1 = 12\sqrt{k_1} = 12\,(34.53)^{0.5} = 70.52,$$

- Lower growth rate near to zero:

$$g = \frac{c_1}{c_0} - 1 = \frac{70.52}{70.47} - 1 = 0.0007.$$

# Utility Level Curves and Production Possibility Curves

$$
\begin{aligned}
u &= \ln(105 - k_1) + (0.98)\ln\left(12\sqrt{k_1}\right), \\
u &= \ln(105 - 34.53) + (0.98)\ln\left(12\sqrt{34.53}\right) = 8.4259, \\
c_1^d &= \left(\frac{e^u}{c_0}\right)^{\frac{1}{\beta}} = \left(\frac{e^{8.4259}}{c_0}\right)^{\frac{1}{0.98}}.
\end{aligned}
$$

$$
c_1^s = 12\sqrt{k_1} = 12\sqrt{105 - c_0}. \tag{43}
$$

# General Equilibrium Graph: Goods Endowment Increase

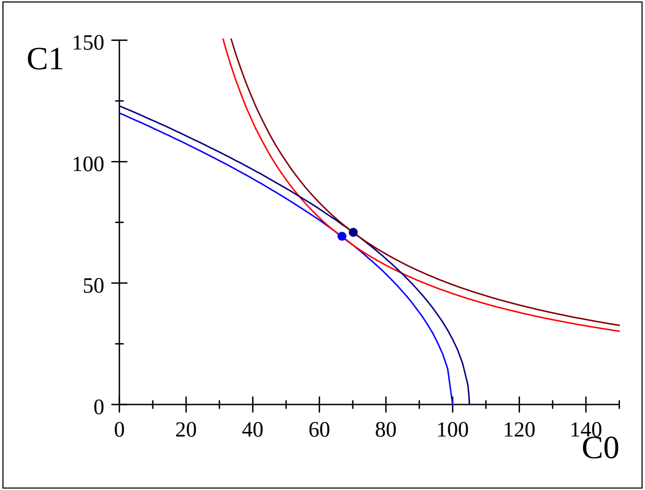


Figure 5.9. General Equilibrium Intertemporal Consumption with Current Income Increase in Example 5.5 (dark red and blue) compared to Example 5.1.

# Decentralized Problem

$$k_1^s = \frac{1}{1.98}\left[105\,(0.98) - \frac{36}{(1+r)^2}\right], \tag{44}$$

$$1+r = \left(\frac{36}{105\,(0.98) - 1.98k_1^s}\right)^{0.5}. \tag{45}$$

$$k_1^s = \frac{1}{1.98}\left[105\,(0.98) - \frac{36}{(1+r)^2}\right] = \frac{36}{(1+r)^2} = k_1^d; \tag{46}$$

$$1+r = \left[\frac{36}{\frac{102.9}{1.98}}\left(1+\frac{1}{1.98}\right)\right]^{0.5} = 1.021; \tag{47}$$

$$k_1^d = \frac{36}{(1+r)^2} = \frac{36}{(1.021)^2} = 34.534. \tag{48}$$

# Capital Market Graph: Current Goods Endowment Increase

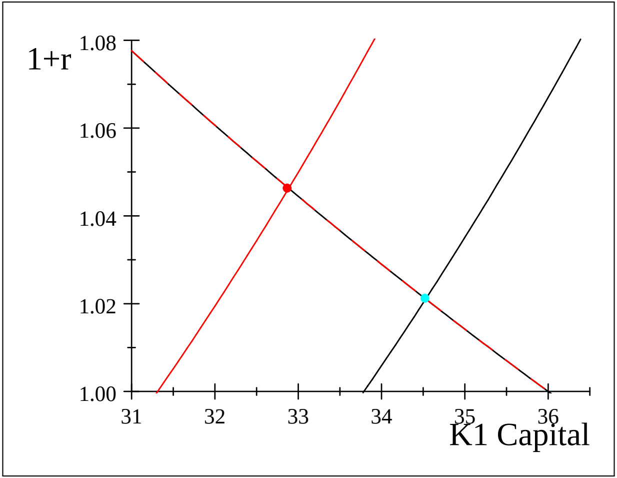


Figure 5.10. An Increase in Current Income Endowment in Example 5.5.

# Future Goods Market

$$c_1^d = \frac{36}{1+r} + \left( \frac{105\,(0.98)}{1.98} - \frac{36}{1.98\,(1+r)^2} \right) (1+r)\,, \qquad (49)$$

$$\frac{1}{1+r} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{c_1^d \pm \sqrt{\left(c_1^d\right)^2 - 4\,(17.82)\,(52)}}{2\,(17.818)}. \qquad (50)$$

$$\frac{1}{1+r} = \frac{c_1^s}{72}. \qquad (51)$$

# Future Goods Market Graph: Output Rises, Interest Rate Falls

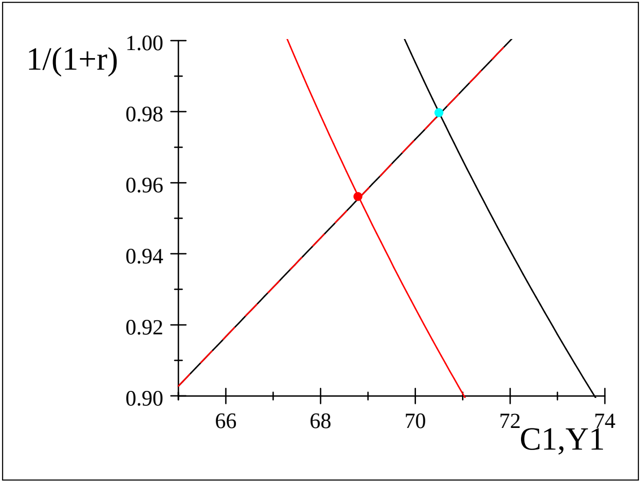


Figure 5.11. Market for Future Period Consumption in Example 5.3.

# General Equilibrium Graph: Add Budget Lines

$$c_1^d = \Pi_1 + k_1^s (1 + r) = \frac{36}{1.021} + (105 - c_0)(1.021);$$

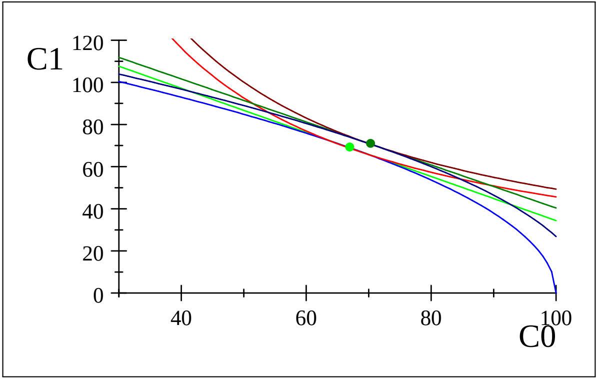


Figure 5.12. General Equilibrium with Income Endowment Increase in Example 5.5 (darker red, blue, green) Compared to Example 5.3 (lighter red, blue, green).

Applications: Savings Rate, Permanent Income, Debt Crisis

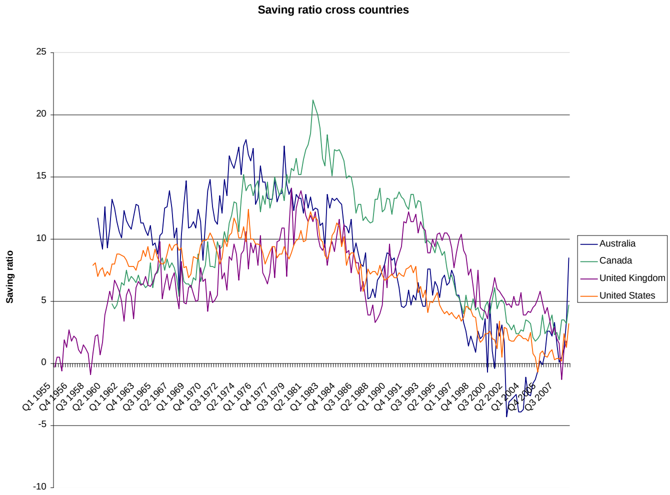


Figure: Figure 5.13. Savings Ratios, 1955-2007, Australia, Canada, UK and US.

# Explaining Low Savings Rates

- Low savings rate policy issue if concern of consequences.
- May be consequence of "wealth effect",
  - a result of rising permanent income stream.
  - Market equity in corporations, homes, shot up in 1990s,
  - seemed that permanent income was rising.
  - As permanent income expected to rise more, less savings needed.
- A theory based on expected permanent income, not current income.
- "the permanent income hypothesis of consumption".
- Implies low savings rate not at a worry in policy sense,
  - if expectations of rising permanent income are realized.
  - Dark underside to recent fall in savings rate:
  - many houses bought by borrowing money: higher expected wealth.
  - Collapse in stock market value 2007-2008 collapsed this wealth.
  - Many houses foreclosed and repossessed by banks.
- Simple economic response to rising wealth, a low savings rate, became urgent policy issue once wealth collapsed and debt could not be repaid.